

Integral structures in automorphic line bundles on the p -adic upper half plane

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Abstract

Given an automorphic line bundle $\mathcal{O}_X(k)$ of weight k on the Drinfel'd upper half plane X over a local field K , we construct a $\mathrm{GL}_2(K)$ -equivariant integral lattice $\mathcal{O}_{\widehat{\mathfrak{X}}}(k)$ in $\mathcal{O}_X(k) \otimes_K \widehat{K}$, as a coherent sheaf on the formal model $\widehat{\mathfrak{X}}$ underlying $X \otimes_K \widehat{K}$. Here \widehat{K}/K is ramified of degree 2. This generalizes a construction of Teitelbaum from the case of even weight k to arbitrary integer weight k . We compute $H^*(\widehat{\mathfrak{X}}, \mathcal{O}_{\widehat{\mathfrak{X}}}(k))$ and obtain applications to the de Rham cohomology $H_{dR}^1(\Gamma \backslash X, \mathrm{Sym}_K^k(\mathrm{St}))$ with coefficients in the k -th symmetric power of the standard representation of $\mathrm{SL}_2(K)$ (where $k \geq 0$) of projective curves $\Gamma \backslash X$ uniformized by X : namely, we prove the degeneration of a certain reduced Hodge spectral sequence computing $H_{dR}^1(\Gamma \backslash X, \mathrm{Sym}_K^k(\mathrm{St}))$, we re-prove the Hodge decomposition of $H_{dR}^1(\Gamma \backslash X, \mathrm{Sym}_K^k(\mathrm{St}))$ and show that the monodromy operator on $H_{dR}^1(\Gamma \backslash X, \mathrm{Sym}_K^k(\mathrm{St}))$ respects integral de Rham structures and is induced by a "universal" monodromy operator defined on $\widehat{\mathfrak{X}}$, i.e. before passing to the Γ -quotient.

Introduction

Let K be a local field and let X be the Drinfel'd upper half plane over K ; that is, the projective line over K with its K -rational points removed. $G = \mathrm{GL}_2(K)$ acts on X . Let $\mathcal{O}_X(k)$ be the structure sheaf on the rigid space X , endowed with the automorphic

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action by G of weight $k \in \mathbb{Z}$. For $k \geq 0$ and even, Teitelbaum [8] constructed a G -invariant integral lattice in $\mathcal{O}_X(k)$, as a line bundle on the natural formal \mathcal{O}_K -scheme \mathfrak{X} underlying X . He then reduced this bundle modulo the maximal ideal of \mathcal{O}_K and determined explicitly its global sections, as a representation of G on an infinite dimensional vector space over the residue field \mathbb{F} of K . The first aim of this paper is to extend his results to *any* weight $k \in \mathbb{Z}$. Now it is not hard to see that for odd k there is no G -equivariant $\mathcal{O}_{\mathfrak{X}}$ -line bundle lattice in $\mathcal{O}_X(k)$. Let \widehat{K} be a ramified extension of K of degree 2, let $\widehat{\mathfrak{X}} = \mathfrak{X} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{K}}$ be the base extended formal $\mathcal{O}_{\widehat{K}}$ -scheme, let $\widetilde{\mathfrak{X}} = \mathfrak{X} \otimes_{\mathcal{O}_K} \mathbb{F} = \widehat{\mathfrak{X}} \otimes_{\mathcal{O}_{\widehat{K}}} \mathbb{F}$. We show that for any $k \in \mathbb{Z}$, if we twist the automorphic action on $\mathcal{O}_X(k)$ by a suitable character, there is a G -equivariant $\mathcal{O}_{\widehat{\mathfrak{X}}}$ -module $\mathcal{O}_{\widehat{\mathfrak{X}}}(k)$ which is a lattice inside $\mathcal{O}_X(k) \otimes_K \widehat{K}$. If k is even it is a line bundle, if k is odd it is not: around the singular points of $\widehat{\mathfrak{X}}$ it needs two generators. We show that $H^0(\widetilde{\mathfrak{X}}, \mathcal{O}_{\widehat{\mathfrak{X}}}(k))$ for $k \geq 0, k \neq 1$ and $H^1(\widetilde{\mathfrak{X}}, \mathcal{O}_{\widehat{\mathfrak{X}}}(k))$ for $k \leq -1$ are *precisely* those cohomology groups which do not vanish. We prove that they are $\mathcal{O}_{\widehat{K}}$ -flat and that their formation commutes with base change to the special fibre $\widetilde{\mathfrak{X}}$. We determine these G -representations obtained by reduction modulo the maximal ideal of $\mathcal{O}_{\widehat{K}}$ in the same manner as in [8]. Next we establish for $k \geq 2$ an isomorphism between $H^0(\widetilde{\mathfrak{X}}, \mathcal{O}_{\widehat{\mathfrak{X}}}(k))$ and a certain space of $\mathcal{O}_{\widehat{K}}$ -module valued harmonic cochains on the Bruhat-Tits tree of G . For $k \geq 2$ and even such an isomorphism was established by analytic methods in [8] whereas we proceed very algebro-geometrically in that we consequently reduce everything modulo the maximal ideal of $\mathcal{O}_{\widehat{K}}$ and work locally on the special fibre $\widetilde{\mathfrak{X}}$. Finally, if $\text{char}(K) = 0$, we demonstrate that integral structures are a strong tool for studying the "reduced" de Rham complex

$$\mathcal{R}_X^\bullet = [\mathcal{O}_X(-k) \xrightarrow{(\frac{d}{dz})^{k+1}} \mathcal{O}_X(k+2)]$$

on X considered in [5], [6], for $k \geq 0$ (here z is a global variable on $X \subset \mathbb{P}_K^1$). It computes the de Rham cohomology $H^*(X, \Omega_X^\bullet \otimes \text{Sym}_K^k(\text{St}))$ of X with coefficients in the k -th symmetric power $\text{Sym}_K^k(\text{St})$ of the standard representation of $\text{SL}_2(K)$. Its differential respects our integral structures, hence a complex

$$\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet = [\mathcal{O}_{\widehat{\mathfrak{X}}}(-k) \xrightarrow{(\frac{d}{dz})^{k+1}} \mathcal{O}_{\widehat{\mathfrak{X}}}(k+2)]$$

on $\widehat{\mathfrak{X}}$. We show that for $k > 0$ we have $H^j(\widetilde{\mathfrak{X}}, \mathcal{R}_{\widetilde{\mathfrak{X}}}^\bullet) = 0$ for $j \neq 1$, while $H^1(\widetilde{\mathfrak{X}}, \mathcal{R}_{\widetilde{\mathfrak{X}}}^\bullet)$ decomposes as

$$H^1(\widetilde{\mathfrak{X}}, \mathcal{R}_{\widetilde{\mathfrak{X}}}^\bullet) \cong H^1(\widetilde{\mathfrak{X}}, \mathcal{O}_{\widehat{\mathfrak{X}}}(-k)) \oplus H^0(\widetilde{\mathfrak{X}}, \mathcal{O}_{\widehat{\mathfrak{X}}}(k+2)) \quad (*).$$

As an application, we show that structural features of the cohomology of varieties uniformized by X can be deduced from (*), thus show up already on X (or rather $\widehat{\mathfrak{X}}$) itself.

Namely we get the well known Hodge decomposition (first obtained by de Shalit [7], see also [5])

$$H_{dR}^1(\Gamma \backslash X, \text{Sym}_K^k(\text{St})) = H^1(\Gamma, \text{Sym}_K^k(\text{St})) \oplus H^0(X_\Gamma, \mathcal{O}_X(k+2)^\Gamma)$$

of $H_{dR}^1(X_\Gamma, \text{Sym}_K^k(\text{St})) = H^1(\Gamma \backslash X, (\Omega_X^\bullet \otimes_K \text{Sym}_K^k(\text{St}))^\Gamma) = H^1(\Gamma \backslash X, (\mathcal{R}_X^\bullet)^\Gamma)$ simply by taking Γ -invariants for a cocompact discrete (torsionfree) subgroup $\Gamma < \text{SL}_2(K)$; no higher Γ -group cohomology is needed. Again, while earlier proofs were truly analytic we reduce everything to algebraic geometry on the irreducible components of $\tilde{\mathfrak{X}}$ (these are all isomorphic to $\mathbb{P}_{\mathbb{F}}^1$). As a bonus of our method we obtain the degeneration of the "reduced" Hodge spectral sequence computing $H_{dR}^1(\Gamma \backslash X, \text{Sym}_K^k(\text{St}))$, as conjectured by Schneider [5], and a complete description (in particular their dimensions) of the cohomology spaces $H^j(\Gamma \backslash X, \mathcal{O}_X(r)^\Gamma)$ (any j, r). Moreover, for $k > 0$, we describe a monodromy operator on $H^1(\tilde{\mathfrak{X}}, \mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet)$ as an isomorphism $H^0(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(k+2)) \cong H^1(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(-k))$. It induces the monodromy operator on $H_{dR}^1(\Gamma \backslash X, \text{Sym}_K^k(\text{St}))$ predicted by p -adic Hodge theory, so in particular we see that the latter respects integral de Rham structures (which in p -adic Hodge theory can not be expected in general) and that its monodromy filtration splits the Hodge filtration.

We mention that the integral structures in $\mathcal{O}_X(k)$ and in the "reduced" de Rham complex considered in this paper play an important role in the recent work of Breuil [1].

Notations: K denotes a non-archimedean locally compact field and K_a its algebraic closure, \mathcal{O}_K its ring of integers, $\pi \in \mathcal{O}_K$ a fixed prime element and \mathbb{F} the residue field with q elements, $q \in p^\mathbb{N}$. We choose $\hat{\pi} \in K_a$ such that $\hat{\pi}^2 = \pi$. Then $\hat{K} = K(\hat{\pi})$ is a ramified extension of K of degree 2 with ring of integers $\mathcal{O}_{\hat{K}}$. We let $\omega : K_a^\times \rightarrow \mathbb{Q}$ be the extension of the discrete valuation $\omega : K^\times \rightarrow \mathbb{Z}$ normalized by $\omega(\pi) = 1$. For formal \mathcal{O}_K -schemes resp. K -rigid spaces we denote by a superscript $\hat{}$ the formal $\mathcal{O}_{\hat{K}}$ -schemes resp. \hat{K} -rigid space obtained by the base change $\mathcal{O}_K \rightarrow \mathcal{O}_{\hat{K}}$ resp. $K \rightarrow \hat{K}$. For $E = K$ or $E = \hat{K}$ and a formal (admissible) \mathcal{O}_E -scheme \mathfrak{W} we let \mathfrak{W}_E be its generic fibre, as a E -rigid space. We need the characters $\chi : G \rightarrow \hat{K}^\times$, $\chi(\gamma) = \hat{\pi}^{\omega(\det \gamma)}$, and $\varepsilon : G \rightarrow \mathcal{O}_K^\times$, $\varepsilon(\gamma) = \pi^{-\omega(\det \gamma)} \det \gamma$, of $G = \text{GL}_2(K)$ and denote the Bruhat-Tits tree of G by \mathcal{BT} . For $r \in \mathbb{R}$ we define $[r], \lceil r \rceil \in \mathbb{Z}$ by requiring $[r] \leq r < [r] + 1$ and $\lceil r \rceil - 1 < r \leq \lceil r \rceil$.

1 Integral structures in automorphic line bundles

Let $X = \Omega_K^{(2)}$ be Drinfel'd's symmetric space of dimension 1 over K . This is the K -rigid space obtained by removing all K -rational points from the projective line \mathbb{P}_K^1 over K . We

choose a coordinate z and define an action of G on X (on the left) by

$$\gamma z = \frac{-b + az}{d - cz} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

([8] takes the other left action). Fix $k \in \mathbb{Z}$. For $f \in \mathcal{O}_{\widehat{X}}$ set

$$(1) \quad f|_{\gamma}(z) = \chi^k(\gamma)(a + cz)^{-k} f\left(\frac{b + dz}{a + cz}\right) \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

Denote by $\mathcal{O}_{\widehat{X}}(k)$ the structure sheaf of the \widehat{K} -rigid space \widehat{X} endowed with the G -action on the left defined by (1). (This is a *left* action; in [8] a right action is considered.)

As explained in [8], the K -rigid space X is the generic fibre of a certain π -adic strictly semistable formal \mathcal{O}_K -scheme \mathfrak{X} : the set F^0 of irreducible components of the reduction $\widetilde{\mathfrak{X}}$ of \mathfrak{X} is in natural bijection with the set of vertices of \mathcal{BT} . Let F^1 be the set of subsets $\{Z_1, Z_2\} \subset F^0$ with $Z_1 \cap Z_2 \neq \emptyset$ and $Z_1 \neq Z_2$; it corresponds to the set of edges of \mathcal{BT} . Each $Z \in F^0$ is isomorphic to $\mathbb{P}_{\mathbb{F}}^1$. The action of G on X extends to \mathfrak{X} . The admissible open subset

$$U = \{P \in \mathbb{P}^1; \quad \omega(z(P)) > -1 \quad \text{and} \quad \omega(z(P) - x) < 1 \quad \text{for all} \quad x \in \mathcal{O}_K\}$$

of X is the tube (=preimage under the specialization map $X \rightarrow \mathfrak{X}$) of the central (with respect to z) irreducible component Z_{γ_0} of $\widetilde{\mathfrak{X}}$. For $\gamma \in G$ define the irreducible component Z_{γ} of $\widetilde{\mathfrak{X}}$ as $Z_{\gamma} = \gamma \cdot Z_{\gamma_0}$. For $n \in \mathbb{Z}$ let

$$\gamma_n = \begin{pmatrix} 1 & 0 \\ 0 & \pi^n \end{pmatrix} \in G.$$

For a subset $E \subset F^0$ let $\widetilde{\mathfrak{U}}_E$ be the maximal open subscheme of $\widetilde{\mathfrak{X}}$ contained in $\cup_{Z \in E} Z$; in other words, the complement in $\widetilde{\mathfrak{X}}$ of the union of all irreducible components not in E . Let \mathfrak{U}_E be the open formal subscheme of \mathfrak{X} lifting $\widetilde{\mathfrak{U}}_E$. Letting

$$\mathfrak{Y} = \mathfrak{U}_{\{Z_{\gamma_n}; n \in \mathbb{Z}\}}$$

we have the open covering

$$\mathfrak{X} = \bigcup_{g \in \text{SL}_2(K)} g \cdot \mathfrak{Y}$$

($\text{SL}_2(K)$ acts transitively on F^1). Let $f_{n,n} \in \mathcal{O}_{\mathfrak{Y}}(\mathfrak{U}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}})$ (resp. $f_{n,n+1} \in \mathcal{O}_{\mathfrak{Y}}(\mathfrak{U}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}})$) be an equation for the closed subscheme $Z_{\gamma_n} \cap \widetilde{\mathfrak{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}$ (resp. $Z_{\gamma_{n+1}} \cap \widetilde{\mathfrak{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}$) of $\mathfrak{U}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}$. In local coordinates, there is an open embedding $\mathfrak{U}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}} \rightarrow \text{Spf}(\mathcal{O}_K <$

$X_1, X_2 > / (X_1 X_2 - \pi)$ such that $f_{n,n} = X_1$ and $f_{n,n+1} = X_2$. Viewing $f_{n,n}$ and $f_{n,n+1}$ as sections of $\mathcal{O}_{\widehat{\mathfrak{Y}}}(\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}})$ we define

$$\mathcal{O}_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}}(k) = \mathcal{O}_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}} \cdot f_{n,n}^{\lceil \frac{kn}{2} \rceil} f_{n,n+1}^{\lceil \frac{k(n+1)}{2} \rceil} + \mathcal{O}_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}} \cdot \widehat{\pi} f_{n,n}^{\lfloor \frac{kn}{2} \rfloor} f_{n,n+1}^{\lfloor \frac{k(n+1)}{2} \rfloor},$$

i.e. the $\mathcal{O}_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}}$ -submodule of $\mathcal{O}_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}} \otimes_{\mathcal{O}_{\widehat{K}}} \widehat{K}$ generated by the two elements $f_{n,n}^{\lceil \frac{kn}{2} \rceil} f_{n,n+1}^{\lceil \frac{k(n+1)}{2} \rceil}$ and $\widehat{\pi} f_{n,n}^{\lfloor \frac{kn}{2} \rfloor} f_{n,n+1}^{\lfloor \frac{k(n+1)}{2} \rfloor}$. If k is even this is just the line bundle generated by the element $z^{\frac{-k}{2}}$. If k is odd this is not a line bundle; an explicit pair of generators is $\widehat{\pi}^{n+1} z^{\frac{-(k-1)}{2}}, \widehat{\pi}^{-n} z^{\frac{-(k+1)}{2}}$.

The $\mathcal{O}_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}}(k)$ glue into an $\mathcal{O}_{\widehat{\mathfrak{Y}}}$ -submodule $\mathcal{O}_{\widehat{\mathfrak{Y}}}(k)$ of $\mathcal{O}_{\widehat{\mathfrak{Y}}} \otimes_{\mathcal{O}_{\widehat{K}}} \widehat{K}$. Note that

$$(2) \quad \mathcal{O}_{\widehat{\mathfrak{Y}}}(k)|_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}}} = \widehat{\pi}^{kn} \mathcal{O}_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}}} \quad \text{inside} \quad \mathcal{O}_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}}} \otimes_{\mathcal{O}_{\widehat{K}}} \widehat{K}.$$

As we remarked, if k is even, $\mathcal{O}_{\widehat{\mathfrak{Y}}}(k)$ is the line bundle generated by the element $z^{\frac{-k}{2}} \in H^0(\widehat{\mathfrak{Y}}, \mathcal{O}_{\widehat{\mathfrak{Y}}} \otimes_{\mathcal{O}_{\widehat{K}}} \widehat{K})$. For any k again we have a canonical identification of sheaves $sp_* \mathcal{O}_{\widehat{X}}(k) = \mathcal{O}_{\widehat{\mathfrak{X}}} \otimes_{\mathcal{O}_{\widehat{K}}} \widehat{K}$ where $sp : \widehat{X} \rightarrow \widehat{\mathfrak{X}}$ is the specialization map; we write $sp_* \mathcal{O}_{\widehat{X}}(k)$ when we refer to the G -equivariant structure on $\mathcal{O}_{\widehat{\mathfrak{X}}} \otimes_{\mathcal{O}_{\widehat{K}}} \widehat{K}$ induced by that on $\mathcal{O}_{\widehat{X}}(k)$.

Proposition 1.1. *Let $\widehat{\mathfrak{W}}, \widehat{\mathfrak{W}}'$ be open formal subschemes of $\widehat{\mathfrak{Y}}$, let $\gamma \in G$ such that $\gamma \widehat{\mathfrak{W}} = \widehat{\mathfrak{W}}'$. Then the isomorphism*

$$\gamma : sp_* \mathcal{O}_{\widehat{X}}(k)|_{\widehat{\mathfrak{W}}} \cong sp_* \mathcal{O}_{\widehat{X}}(k)|_{\widehat{\mathfrak{W}}'}$$

induces an isomorphism of subsheaves

$$\gamma : \mathcal{O}_{\widehat{\mathfrak{Y}}}(k)|_{\widehat{\mathfrak{W}}} \cong \mathcal{O}_{\widehat{\mathfrak{Y}}}(k)|_{\widehat{\mathfrak{W}}}.$$

PROOF: (a) First we assume $\widehat{\mathfrak{W}} \subset \widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}}$ for some n ; then also $\widehat{\mathfrak{W}}' \subset \widehat{\mathfrak{U}}_{\{Z_{\gamma_{n'}}\}}$ for some n' and (2) applies to $\widehat{\mathfrak{W}}$ and $\widehat{\mathfrak{W}}'$. In that situation we must show

$$(3) \quad 2\omega((a + cz(P))^{-k}) + k\omega(ad - bc) = k(n' - n) \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for each point P in the generic fibre $\widehat{\mathfrak{W}}'_{\widehat{K}}$ of $\widehat{\mathfrak{W}}'$. Note that $\gamma Z_{\gamma_n} = Z_{\gamma_{n'}}$ and thus $\gamma_{n'}^{-1} \gamma \gamma_n$ stabilizes Z_{γ_0} , hence is an element of $K^\times \cdot \text{GL}_2(\mathcal{O}_K)$; in other words, $\gamma = \gamma_{n'} \delta \gamma_n^{-1}$ for some $\delta \in K^\times \cdot \text{GL}_2(\mathcal{O}_K)$. Therefore it suffices to check (3) in the cases

- (i) $\gamma = \gamma_m$ and $n' = n + m$ for some $m \in \mathbb{Z}$;
- (ii) $b = c = 0 = n = n'$ and $a = d$;
- (iii) $n = n' = 0$ and $\gamma \in \text{GL}_2(\mathcal{O}_K)$.

In either case (3) is immediate; for the case (iii) note that $\omega(z(P) - \beta) = 0$ for any $\beta \in \mathcal{O}_K^\times$.

(b) Now let $\widehat{\mathfrak{W}}, \widehat{\mathfrak{W}'}$ be arbitrary. By construction, both $\mathcal{L}_1 = \gamma_*(\mathcal{O}_{\widehat{\mathfrak{Y}}}(k)|_{\widehat{\mathfrak{W}}})$ and $\mathcal{L}_2 = \mathcal{O}_{\widehat{\mathfrak{Y}}}(k)|_{\widehat{\mathfrak{W}'}}$ are $\mathcal{O}_{\widehat{\mathfrak{W}'}}$ -modules contained in $\mathcal{O}_{\widehat{\mathfrak{W}'}} \otimes_{\mathcal{O}_{\widehat{K}}} \widehat{K}$ as lattices, i.e. $\mathcal{L}_i \otimes_{\mathcal{O}_{\widehat{K}}} \widehat{K} = \mathcal{O}_{\widehat{\mathfrak{W}'}} \otimes_{\mathcal{O}_{\widehat{K}}} \widehat{K}$. By (a) we have $\mathcal{L}_1|_{\widehat{\mathfrak{Y}}} = \mathcal{L}_2|_{\widehat{\mathfrak{Y}}}$ for an open formal subscheme $\widehat{\mathfrak{Y}}$ of $\widehat{\mathfrak{W}'}$ whose reduction is dense in the reduction of $\widehat{\mathfrak{W}'}$. All this implies $\mathcal{L}_1 = \mathcal{L}_2$, using the following fact: for open formal subschemes $\widehat{\mathfrak{Y}}_1 \subset \widehat{\mathfrak{Y}}_2$ of $\widehat{\mathfrak{Y}}$ with $\widehat{\mathfrak{Y}}_1$ dense in $\widehat{\mathfrak{Y}}_2$, and for $f \in (\mathcal{O}_{\widehat{\mathfrak{Y}}}(k) \otimes_{\mathcal{O}_{\widehat{K}}} \widehat{K})(\widehat{\mathfrak{Y}}_2)$ we have $f \in \mathcal{O}_{\widehat{\mathfrak{Y}}}(k)(\widehat{\mathfrak{Y}}_2)$ if and only if $f \in \mathcal{O}_{\widehat{\mathfrak{Y}}}(k)(\widehat{\mathfrak{Y}}_1)$. To see this fact it suffices to show that for $g \in (\mathcal{O}_{\widehat{\mathfrak{Y}}}(k)/(\widehat{\pi}))(\widehat{\mathfrak{Y}}_2)$ we have $g = 0$ if and only if $g|_{\widehat{\mathfrak{Y}}_1} = 0$ in $(\mathcal{O}_{\widehat{\mathfrak{Y}}}(k)/(\widehat{\pi}))(\widehat{\mathfrak{Y}}_1)$. This is immediate from the local analysis in section 2 below. \square

Thanks to 1.1 we can now move around $\mathcal{O}_{\widehat{\mathfrak{Y}}}(k)$ by means of the G -action on $\widehat{\mathfrak{X}}$ and obtain a G -equivariant coherent $\mathcal{O}_{\widehat{\mathfrak{X}}}$ -module lattice $\mathcal{O}_{\widehat{\mathfrak{X}}}(k)$ inside $sp_*\mathcal{O}_{\widehat{X}}(k)$.

For $k_1, k_2 \in \mathbb{Z}$ we have a G -equivariant surjective map (not needed in the sequel)

$$\mathcal{O}_{\widehat{\mathfrak{X}}}(k_1) \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_{\widehat{\mathfrak{X}}}(k_2) \longrightarrow \mathcal{O}_{\widehat{\mathfrak{X}}}(k_1 + k_2)$$

which is multiplication of functions. This follows from equation (2) and the argument in part (b) of the proof of 1.1. It is an isomorphism if at least one of k_1 or k_2 is even, for in that case we are tensoring with a line bundle. On the other hand, it cannot be an isomorphism if both k_1 and k_2 are odd, because then the fibres of both $\mathcal{O}_{\widehat{\mathfrak{X}}}(k_j)$ at singular points of \widetilde{X} are 2-dimensional, whereas $\mathcal{O}_{\widehat{\mathfrak{X}}}(k_1 + k_2)$ is a line bundle (in this case).

2 Cohomology

For divisors D on $\mathbb{P}_{\mathbb{F}}^1$ let $\mathcal{L}(D)$ be the corresponding line bundle on $\mathbb{P}_{\mathbb{F}}^1$. By the usual convention, $\mathcal{L}(-D) \subset \mathcal{O}_{\mathbb{P}_{\mathbb{F}}^1}$ if D is an effective divisor. Fix a system R of representatives for \mathbb{F} in \mathcal{O}_K . For $a \in R$ and $n \in \mathbb{Z}$ let

$$\gamma_{a,n} = \begin{pmatrix} 1 & \pi^{-n}a \\ 0 & 1 \end{pmatrix}.$$

An easy consideration on \mathcal{BT} shows that

$$\{Z_{\gamma_{n+1}}\} \bigcup \{Z_{\gamma_{a,n}\gamma_{n-1}}; a \in R\}$$

is the set of the $q + 1$ many irreducible components of $\widetilde{\mathfrak{X}}$ meeting Z_{γ_n} . (The function $\pi^{n-1}z + \pi^{-1}a$ is a coordinate on $\widehat{\mathfrak{U}}_{\{Z_{\gamma_{a,n}\gamma_{n-1}}\}}$ in the sense that $\omega(\pi^{n-1}z(P) + \pi^{-1}a) = 0$ for any $P \in (\widehat{\mathfrak{U}}_{\{Z_{\gamma_{a,n}\gamma_{n-1}}\}})_{\widehat{K}}$.) Since $\gamma_{a,n}$ acts on $sp_*\mathcal{O}_{\widehat{X}}(k)$ with trivial automorphy factor it induces an isomorphism

$$\gamma_{a,n} : \widehat{\pi}^{k(n-1)}\mathcal{O}_{\widehat{\mathfrak{X}}}(k)|_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_{n-1}}, Z_{\gamma_n}\}}} \cong \widehat{\pi}^{k(n-1)}\mathcal{O}_{\widehat{\mathfrak{X}}}(k)|_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_{a,n}\gamma_{n-1}}, Z_{\gamma_n}\}}}.$$

Using this we can now give a local description of the G -equivariant coherent $\mathcal{O}_{\tilde{\mathfrak{X}}}$ -module $\mathcal{O}_{\tilde{\mathfrak{X}}}(k)/(\hat{\pi})$ which we denote by $\mathcal{O}_{\tilde{\mathfrak{X}}}(k)$.

(a) *First assume that k is even.* Let $h_a \in \mathcal{O}_{Z_{\gamma_n}}$ be a local equation for $Z_{\gamma_n} \cap Z_{\gamma_{a,n}\gamma_{n-1}}$ in Z_{γ_n} , let $h_{\infty} \in \mathcal{O}_{Z_{\gamma_n}}$ be a local equation for $Z_{\gamma_n} \cap Z_{\gamma_{n+1}}$ in Z_{γ_n} . Then $\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_n}}$ is isomorphic to the following $\mathcal{O}_{Z_{\gamma_n}}$ -submodule of the constant "rational function field" sheaf on $Z_{\gamma_n} \cong \mathbb{P}_{\mathbb{F}}^1$: locally around $Z_{\gamma_n} \cap Z_{\gamma_{a,n}\gamma_{n-1}}$ it is generated by $h_a^{\frac{k(n-1)}{2} - \frac{kn}{2}}$, locally around $Z_{\gamma_n} \cap Z_{\gamma_{n+1}}$ it is generated by $h_{\infty}^{\frac{k(n+1)}{2} - \frac{kn}{2}}$, and locally around other points it coincides with $\mathcal{O}_{Z_{\gamma_n}}$. Thus

$$(4) \quad \mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z \cong \mathcal{L}\left(\frac{-k}{2} \cdot \infty + \sum_{b \in \mathbb{F}} \frac{k}{2} \cdot b\right)$$

for $Z = Z_{\gamma_n}$. By equivariance we get (4) for any $Z \in F^0$. In particular, $\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z$ is of degree $\frac{(q-1)k}{2}$.

(b) *Now assume that k is odd.* For $Z \in F^0$ let

$$(\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z)^c = \frac{\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z}{(\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z)_{\text{torsion}}}.$$

We then have

$$\mathcal{O}_{\tilde{\mathfrak{X}}}(k)|_{\tilde{\mathfrak{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}} = (\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_n}})^c|_{\tilde{\mathfrak{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}} \oplus (\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_{n+1}}})^c|_{\tilde{\mathfrak{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}}.$$

Explicitly, $(\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_n}})^c|_{\tilde{\mathfrak{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}}$ is generated by $f_{n,n}^{\frac{kn}{2}} f_{n,n+1}^{\frac{k(n+1)+1}{2}}$ if n is even, and by $\hat{\pi} f_{n,n}^{\frac{kn-1}{2}} f_{n,n+1}^{\frac{k(n+1)}{2}}$ if n is odd. $(\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_{n+1}}})^c|_{\tilde{\mathfrak{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}}$ is generated by $\hat{\pi} f_{n,n}^{\frac{kn}{2}} f_{n,n+1}^{\frac{k(n+1)-1}{2}}$ if n is even, and by $f_{n,n}^{\frac{kn+1}{2}} f_{n,n+1}^{\frac{k(n+1)}{2}}$ if n is odd. Now we proceed as in (a). By what we just saw, $(\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_n}})^c$ is generated around $Z_{\gamma_{n+1}} \cap Z_{\gamma_n}$ by $h_{\infty}^{\lceil \frac{kn}{2} \rceil - \lceil \frac{k(n+1)}{2} \rceil}$, and around $Z_{\gamma_n} \cap Z_{\gamma_{a,n}\gamma_{n-1}}$ by $h_a^{\lceil \frac{k(n-1)}{2} \rceil - \lceil \frac{kn}{2} \rceil}$ (by equivariance, it suffices to check the latter for $a = 0$). Thus

$$(5) \quad \mathcal{O}_{\tilde{\mathfrak{X}}}(k) = \prod_{Z \in F^0} (\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z)^c,$$

$$(6) \quad (\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z)^c \cong \mathcal{L}\left(\frac{-k-1}{2} \cdot \infty + \sum_{b \in \mathbb{F}} \frac{k-1}{2} \cdot b\right)$$

for $Z = Z_{\gamma_n}$. By equivariance we get (6) for any $Z \in F^0$. In particular, $(\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z)^c$ is of degree $\frac{(q-1)(k-1)}{2} - 1$.

Theorem 2.1. (a) $H^*(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k))$ is $\mathcal{O}_{\hat{K}}$ -flat and

$$H^*(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k)) = H^*(\tilde{\mathcal{X}}, \mathcal{O}_{\hat{\mathcal{X}}}(k))/(\pi).$$

(b) For $k \leq -1$ and also for $k = 1$ we have $H^0(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k)) = 0$.

(c) For $k \geq 0$ we have $H^1(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k)) = 0$.

PROOF: (i) First assume k is even. To prove (c) it is enough to prove

$$(7) \quad \mathbb{R}^1 \lim_{\leftarrow t} H^0(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k)/(\hat{\pi}^t)) = 0$$

$$(8) \quad \lim_{\leftarrow t} H^1(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k)/(\hat{\pi}^t)) = 0.$$

For (7) it suffices to show surjectivity of all transition maps $H^0(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k)/(\hat{\pi}^{t+1})) \rightarrow H^0(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k)/(\hat{\pi}^t))$. Using the long exact cohomology sequence associated with

$$(9) \quad 0 \rightarrow \mathcal{O}_{\tilde{\mathcal{X}}}(k) \xrightarrow{\hat{\pi}^t} \mathcal{O}_{\tilde{\mathcal{X}}}(k)/(\hat{\pi}^{t+1}) \rightarrow \mathcal{O}_{\tilde{\mathcal{X}}}(k)/(\hat{\pi}^t) \rightarrow 0$$

this will be implied by

$$(10) \quad H^1(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k)) = 0.$$

Also (8) is reduced to (10) using (9), so let us prove (10). We have an exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{\mathcal{X}}}(k) \rightarrow \prod_{Z \in F^0} \mathcal{O}_{\tilde{\mathcal{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathcal{X}}}} \mathcal{O}_Z \rightarrow \prod_{\{Z_1, Z_2\} \in F^1} \mathcal{O}_{\tilde{\mathcal{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathcal{X}}}} \mathcal{O}_{Z_1 \cap Z_2} \rightarrow 0$$

and a corresponding long exact sequence in cohomology. We know

$$H^1(\tilde{\mathcal{X}}, \prod_{Z \in F^0} \mathcal{O}_{\tilde{\mathcal{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathcal{X}}}} \mathcal{O}_Z) = \prod_{Z \in F^0} H^1(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathcal{X}}}} \mathcal{O}_Z) = 0$$

because $\mathcal{O}_{\tilde{\mathcal{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathcal{X}}}} \mathcal{O}_Z$ is isomorphic to a line bundle on $\mathbb{P}_{\mathbb{F}}^1 \cong Z$ of non-negative degree as we saw above (since $k \geq 0$). On the other hand

$$H^0(\tilde{\mathcal{X}}, \prod_{Z \in F^0} \mathcal{O}_{\tilde{\mathcal{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathcal{X}}}} \mathcal{O}_Z) \rightarrow H^0(\tilde{\mathcal{X}}, \prod_{\{Z_1, Z_2\} \in F^1} \mathcal{O}_{\tilde{\mathcal{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathcal{X}}}} \mathcal{O}_{Z_1 \cap Z_2})$$

is surjective: This follows from the contractibility of \mathcal{BT} and again the fact that each $\mathcal{O}_{\tilde{\mathcal{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathcal{X}}}} \mathcal{O}_Z$ has non-negative degree, which implies that

$$H^0(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathcal{X}}}} \mathcal{O}_{Z_1}) \rightarrow H^0(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathcal{X}}}} \mathcal{O}_{Z_1 \cap Z_2})$$

for any $\{Z_1, Z_2\} \in F^1$ is surjective. To prove (b), since $H^0(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k)) = \lim_{\leftarrow t} H^0(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k)/(\pi^t))$ we can reduce, using the long exact cohomology sequence associated with (9), to the statement

$$H^0(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k)) = 0.$$

But this follows immediately from the injectivity of

$$H^0(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(k)) \longrightarrow H^0(\tilde{\mathfrak{X}}, \prod_{Z \in F^0} \mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z)$$

and the fact that $\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z$ for each $Z \in F^0$ is isomorphic to a line bundle on $\mathbb{P}_{\mathbb{F}}^1 \cong Z$ of negative degree as we saw above. To see the $\mathcal{O}_{\hat{K}}$ -flatness of $H^*(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(k))$ in (a) we need to show injectivity of multiplication with $\hat{\pi}$. This follows from (the proof of) (b) and (c) and the long exact cohomology sequence associated with

$$0 \longrightarrow \mathcal{O}_{\hat{\mathfrak{X}}}(k) \xrightarrow{\hat{\pi}} \mathcal{O}_{\tilde{\mathfrak{X}}}(k) \longrightarrow \mathcal{O}_{\tilde{\mathfrak{X}}}(k) \longrightarrow 0.$$

The base change statement follows similarly.

(ii) For odd k the proofs are similar but easier in view of the decomposition (5). \square

The important vanishing $H^1(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(k)) = 0$ was asserted for even $k \geq 0$ in [8] Cor.24. However, the comparison with $H^1(\hat{X}, \mathcal{O}_{\hat{X}}(k))$ invoked there does not seem to be justified.

Let $\Gamma < \mathrm{SL}_2(K)$ be a cocompact discrete subgroup which for simplicity we assume to be torsion free (in general it contains a torsion free subgroup of finite index). Let $X_\Gamma = \Gamma \backslash X$, $\hat{X}_\Gamma = \Gamma \backslash \hat{X}$, $\tilde{\mathfrak{X}}_\Gamma = \Gamma \backslash \tilde{\mathfrak{X}}$ and $\tilde{\mathfrak{X}}_\Gamma = \Gamma \backslash \tilde{\mathfrak{X}}$ be the quotients for the free action by Γ ; they all algebraize to projective schemes.

Corollary 2.2. (a) For $k > 0$ we have

$$H^0(\tilde{\mathfrak{X}}_\Gamma, \mathcal{O}_{\tilde{\mathfrak{X}}_\Gamma}(-k)^\Gamma) = 0 = H^1(\tilde{\mathfrak{X}}_\Gamma, \mathcal{O}_{\tilde{\mathfrak{X}}_\Gamma}(k+2)^\Gamma).$$

In particular, $H^0(X_\Gamma, \mathcal{O}_X(-k)^\Gamma) = 0 = H^1(X_\Gamma, \mathcal{O}_X(k+2)^\Gamma)$.

(b) $H^0(\tilde{\mathfrak{X}}_\Gamma, \mathcal{O}_{\tilde{\mathfrak{X}}_\Gamma}(k+2)^\Gamma)$ and $H^1(\tilde{\mathfrak{X}}_\Gamma, \mathcal{O}_{\tilde{\mathfrak{X}}_\Gamma}(-k)^\Gamma)$ are $\mathcal{O}_{\hat{K}}$ -flat and

$$H^0(\tilde{\mathfrak{X}}_\Gamma, \mathcal{O}_{\tilde{\mathfrak{X}}_\Gamma}(k+2)^\Gamma) \otimes_{\mathcal{O}_{\hat{K}}} \mathbb{F} = H^0(\tilde{\mathfrak{X}}_\Gamma, \mathcal{O}_{\tilde{\mathfrak{X}}_\Gamma}(k+2)^\Gamma)$$

$$H^1(\tilde{\mathfrak{X}}_\Gamma, \mathcal{O}_{\tilde{\mathfrak{X}}_\Gamma}(-k)^\Gamma) \otimes_{\mathcal{O}_{\hat{K}}} \mathbb{F} = H^1(\tilde{\mathfrak{X}}_\Gamma, \mathcal{O}_{\tilde{\mathfrak{X}}_\Gamma}(-k)^\Gamma).$$

(c) Serre duality identifies $H^1(X_\Gamma, \mathcal{O}_X(-k)^\Gamma)$ with the dual of $H^0(X_\Gamma, \mathcal{O}_X(k+2)^\Gamma)$.

(d)

$$H^j(\tilde{\mathfrak{X}}_\Gamma, \mathcal{O}_{\tilde{\mathfrak{X}}_\Gamma}(1)^\Gamma) = 0$$

for any j . In particular, $H^j(X_\Gamma, \mathcal{O}_X(1)^\Gamma) = 0$.

PROOF: (a) For odd k literally the same proof as in 2.1 applies, because in that case we have the decomposition (5) which allows us to reduce to problems on each irreducible component — these are the same for $\tilde{\mathfrak{X}}$ and $\tilde{\mathfrak{X}}_\Gamma$. Now let k be even. From 2.1 we

get $H^0(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(-k)) = 0$ and $H^0(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(-k)) = 0$. In particular $H^0(\tilde{\mathfrak{X}}_\Gamma, \mathcal{O}_{\tilde{\mathfrak{X}}}(-k)^\Gamma) = H^0(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(-k))^\Gamma = 0$ and $H^0(\tilde{\mathfrak{X}}_\Gamma, \mathcal{O}_{\tilde{\mathfrak{X}}}(-k)^\Gamma) = H^0(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(-k))^\Gamma = 0$. Now we have a $\mathrm{SL}_2(K)$ -equivariant isomorphism $\mathcal{O}_{\tilde{\mathfrak{X}}}(2) \cong \Omega_{\tilde{\mathfrak{X}}}^1$ on $\tilde{\mathfrak{X}}$, where $\Omega_{\tilde{\mathfrak{X}}}^1$ is the sheaf of relative logarithmic differentials for the log smooth formal $\mathrm{Spf}(\mathcal{O}_{\hat{K}})$ -scheme $\hat{\mathfrak{X}}$ (with respect to the pull back log structures from the canonical log structures on \mathfrak{X} and $\mathrm{Spf}(\mathcal{O}_K)$). Thus $\mathcal{O}_{\tilde{\mathfrak{X}}}(2)^\Gamma$ can be identified with the sheaf of relative logarithmic differentials for the log smooth projective $\mathrm{Spec}(\mathcal{O}_{\hat{K}})$ -scheme $\hat{\mathfrak{X}}_\Gamma$. This is a dualizing sheaf by [3] ch.I, sect.2, where it is called the sheaf of *regular* differentials (the generalization to general projective log schemes is [9] Theorem 2.21). Since $\mathcal{O}_{\tilde{\mathfrak{X}}}(k+2)^\Gamma = (\mathcal{O}_{\tilde{\mathfrak{X}}}(-k)^\Gamma)^{\otimes(-1)} \otimes \mathcal{O}_{\tilde{\mathfrak{X}}}(2)^\Gamma$ (note that since k is even we are dealing with line bundles here) we get $H^1(\tilde{\mathfrak{X}}_\Gamma, \mathcal{O}_{\tilde{\mathfrak{X}}}(k+2)^\Gamma) = 0$ by Serre duality. The same argument works for the sheaves $\mathcal{O}_{\tilde{\mathfrak{X}}}(\cdot)$. For (b) we may now proceed as in 2.1. For (c) note that $\mathcal{O}_X(2)$ is $\mathrm{SL}_2(K)$ -equivariantly isomorphic with the sheaf Ω_X^1 of differentials on X , hence $\mathcal{O}_X(k+2)^\Gamma \cong (\mathcal{O}_X(-k)^\Gamma)^{\otimes(-1)} \otimes \Omega_{X^\Gamma}^1$ (for even k we just saw the integral version in (a)). The statements in (d) follow immediately from 2.1. \square

The fact $H^0(X_\Gamma, \mathcal{O}_X(1)^\Gamma) = 0$ ("there are no non zero automorphic forms for Γ of weight one") was proven by analytic methods in [6] Cor.13. For the K -vector space dimensions of $H^1(X_\Gamma, \mathcal{O}_X(-k)^\Gamma)$ and of $H^0(X_\Gamma, \mathcal{O}_X(k+2)^\Gamma)$ see 5.3 below.

3 Modular representations

Denote by $\mathcal{I} \subset \mathcal{O}_{\tilde{\mathfrak{X}}}$ the ideal sheaf of functions vanishing at the singular points of $\tilde{\mathfrak{X}}$. For $k \in \mathbb{Z}$ and $i \geq 0$ let

$$\mathcal{O}_{\tilde{\mathfrak{X}}}(k)(i) = \mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{I}^i.$$

Let $Z \in F^0$. If k is odd we let

$$(\mathcal{O}_{\tilde{\mathfrak{X}}}(k)(i) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z)^c = (\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z)^c \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{I}^i.$$

To unify notations, if k is even we let $(\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z)^c = \mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z$ and

$$(\mathcal{O}_{\tilde{\mathfrak{X}}}(k)(i) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z)^c = \mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{I}^i,$$

i.e. for even k the outer $(\cdot)^c$ is redundant. We have

$$(11) \quad \mathcal{O}_{\tilde{\mathfrak{X}}}(k)(i) = \prod_{Z \in F^0} (\mathcal{O}_{\tilde{\mathfrak{X}}}(k)(i) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z)^c.$$

if k is odd and $i \geq 0$ arbitrary, and also if k is even and $i > 0$. In particular, for such (k, i) we have for any $Z \in F^0$ the natural injection

$$\iota_Z : (\mathcal{O}_{\tilde{\mathfrak{X}}}(k)(i) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z)^c \longrightarrow \mathcal{O}_{\tilde{\mathfrak{X}}}(k)(i)$$

and the canonical projection map

$$\rho_Z : \mathcal{O}_{\tilde{\mathfrak{X}}}(k)(i) \longrightarrow (\mathcal{O}_{\tilde{\mathfrak{X}}}(k)(i) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z)^c.$$

We denote maps induced by ι_Z resp. ρ_Z in cohomology again by ι_Z resp. ρ_Z .

Lemma 3.1. *Suppose $i \geq 0$ if k is odd, or $i > 0$ if k is even. Then we have a canonical G -equivariant isomorphism*

$$H^*(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(k)(i)) \cong \text{Ind}_{K^\times \text{GL}_2(\mathcal{O}_K)}^G H^*(\tilde{\mathfrak{X}}, (\mathcal{O}_{\tilde{\mathfrak{X}}}(k)(i) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c)$$

PROOF: By definition, $\text{Ind}_{K^\times \text{GL}_2(\mathcal{O}_K)}^G H^*(\tilde{\mathfrak{X}}, (\mathcal{O}_{\tilde{\mathfrak{X}}}(k)(i) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c)$ is the space of locally constant functions $u : G \rightarrow H^*(\tilde{\mathfrak{X}}, (\mathcal{O}_{\tilde{\mathfrak{X}}}(k)(i) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c)$ which satisfy $u(\eta\gamma) = \eta(u(\gamma))$ for $\eta \in K^\times \text{GL}_2(\mathcal{O}_K)$, $\gamma \in G$. The action of G is by $(\gamma.u)(\gamma') = u(\gamma'\gamma)$. Note that $K^\times \text{GL}_2(\mathcal{O}_K)$ is the stabilizer of Z_{γ_0} in G . Let $S \subset G$ be a subset such that $\gamma \mapsto Z_\gamma$ is a bijection between S and F^0 . The desired map is

$$f \mapsto [u : G \rightarrow H^*(\tilde{\mathfrak{X}}, (\mathcal{O}_{\tilde{\mathfrak{X}}}(k)(i) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c), \gamma \mapsto \rho_{Z_{\gamma_0}}(\gamma.f)].$$

Its inverse is

$$[u : G \rightarrow H^*(\tilde{\mathfrak{X}}, (\mathcal{O}_{\tilde{\mathfrak{X}}}(k)(i) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c)] \mapsto \sum_{\gamma \in S} \gamma(\iota_{Z_{\gamma_0}}(u(\gamma^{-1}))).$$

Note that $\gamma(\iota_{Z_{\gamma_0}}(u(\gamma^{-1})))$ is supported only on Z_γ . □

For a commutative ring A and integers n, s with $n \geq 0$ let us denote by $\text{Sym}_A^n(\text{St})[s]$ the free A -module of homogeneous polynomials $F(X, Y)$ of degree n in the variables X, Y with coefficients in A , together with its $\text{GL}_2(A)$ -action

$$\gamma.F(X, Y) = (ad - bc)^s F(dX + bY, cX + aY) \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(A).$$

Now consider for $k \in \mathbb{Z}$ the action of $\text{GL}_2(\mathbb{F})$ on $\mathbb{F}(z)$ given by

$$(12) \quad f|_\gamma(z) = \left(\frac{1}{a + cz}\right)^k f\left(\frac{b + dz}{a + cz}\right) \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We view $\mathbb{F}(z)$ as the function field of $\mathbb{P}_{\mathbb{F}}^1 = \text{Spec}(\mathbb{F}[z]) \cup \{\infty\}$ and will consider line bundles on $\mathbb{P}_{\mathbb{F}}^1$ stable for (12). Let $i \geq 0$.

Lemma 3.2. (a) *Suppose k is even and $t = \frac{(q-1)k}{2} - i(q+1) \geq 0$. Then, as $\text{GL}_2(\mathbb{F})$ -representations,*

$$\text{Sym}_{\mathbb{F}}^t(\text{St})[i - \frac{k}{2}] \cong H^0(\mathbb{P}_{\mathbb{F}}^1, \mathcal{L}(\sum_{b \in \mathbb{F}} (\frac{k}{2} - i).b - (\frac{k}{2} + i).\infty)).$$

(b) Suppose k is odd and $t = \frac{(q-1)k-(q+1)}{2} - i(q+1) \geq 0$. Then, as $\mathrm{GL}_2(\mathbb{F})$ -representations,

$$\mathrm{Sym}_{\mathbb{F}}^t(\mathrm{St})[i - \frac{k-1}{2}] \cong H^0(\mathbb{P}_{\mathbb{F}}^1, \mathcal{L}(\sum_{b \in \mathbb{F}} (\frac{k-1}{2} - i).b - (\frac{k+1}{2} + i).\infty)).$$

PROOF: In (a) the map sends $X^r Y^{t-r}$ to $z^r(z - z^q)^{i-\frac{k}{2}}$ for $0 \leq r \leq t$. In (b) it sends $X^r Y^{t-r}$ to $z^r(z - z^q)^{i-\frac{k-1}{2}}$. \square

We view $\mathrm{Sym}_{\mathbb{F}}^n(\mathrm{St})[s]$ as a $\mathrm{GL}_2(\mathcal{O}_K)$ -representation via the canonical map $\mathrm{GL}_2(\mathcal{O}_K) \rightarrow \mathrm{GL}_2(\mathbb{F})$, and we then extend the action further to an action by $K^\times \mathrm{GL}_2(\mathcal{O}_K)$ by sending $\pi \in K^\times$ (i.e. the diagonal matrix with both entries equal to $\pi = \widehat{\pi}^2$) to the identity.

Theorem 3.3. (a) Suppose k is even, $i > 0$ and $t = \frac{(q-1)k}{2} - i(q+1) \geq 0$. Then we have a canonical G -equivariant isomorphism

$$H^0(\widetilde{\mathfrak{X}}, \mathcal{O}_{\widetilde{\mathfrak{X}}}(k)(i)) \cong \mathrm{Ind}_{K^\times \mathrm{GL}_2(\mathcal{O}_K)}^G \mathrm{Sym}_{\mathbb{F}}^t(\mathrm{St})[i - \frac{k}{2}].$$

(b) Suppose k is odd, $i \geq 0$ and $t = \frac{(q-1)(k-1)}{2} - 1 - i(q+1) \geq 0$. Then we have a canonical G -equivariant isomorphism

$$H^0(\widetilde{\mathfrak{X}}, \mathcal{O}_{\widetilde{\mathfrak{X}}}(k)(i)) \cong \mathrm{Ind}_{K^\times \mathrm{GL}_2(\mathcal{O}_K)}^G \mathrm{Sym}_{\mathbb{F}}^t(\mathrm{St})[i - \frac{k-1}{2}].$$

PROOF: We lift the $\mathrm{GL}_2(\mathbb{F})$ -action on $H^0(\mathbb{P}_{\mathbb{F}}^1, \mathcal{L}(\sum_{b \in \mathbb{F}} (\frac{k}{2} - i).b - (\frac{k}{2} + i).\infty))$ if k is even, resp. on $H^0(\mathbb{P}_{\mathbb{F}}^1, \mathcal{L}(\sum_{b \in \mathbb{F}} (\frac{k-1}{2} - i).b - (\frac{k+1}{2} + i).\infty))$ if k is odd, to an action by $K^\times \mathrm{GL}_2(\mathcal{O}_K)$ in the same way as explained for $\mathrm{Sym}_{\mathbb{F}}^n(\mathrm{St})[s]$. Identifying the reduction of the global variable z with our projective coordinate z on $Z_{\gamma_0} \cong \mathbb{P}_{\mathbb{F}}^1$ we use (6) and (4) to get $K^\times \mathrm{GL}_2(\mathcal{O}_K)$ -equivariant isomorphisms

$$H^0(\mathbb{P}_{\mathbb{F}}^1, \mathcal{L}(\sum_{b \in \mathbb{F}} (\frac{k}{2} - i).b - (\frac{k}{2} + i).\infty)) = H^0(\widetilde{\mathfrak{X}}, (\mathcal{O}_{\widetilde{\mathfrak{X}}}(k)(i) \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c)$$

if k is even, resp.

$$H^0(\mathbb{P}_{\mathbb{F}}^1, \mathcal{L}(\sum_{b \in \mathbb{F}} (\frac{k-1}{2} - i).b - (\frac{k+1}{2} + i).\infty)) = H^0(\widetilde{\mathfrak{X}}, (\mathcal{O}_{\widetilde{\mathfrak{X}}}(k)(i) \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c)$$

if k is odd, thus we conclude by 3.1 and 3.2. \square

We can now filter the representation $H^0(\widetilde{\mathfrak{X}}, \mathcal{O}_{\widetilde{\mathfrak{X}}}(k))$ and determine its subquotients. For k odd, $i \geq 0$ and $t = \frac{(q-1)(k-1)}{2} - 1 - i(q+1) \geq q+1$ we have $H^1(\widetilde{\mathfrak{X}}, (\mathcal{O}_{\widetilde{\mathfrak{X}}}(k)(i+1) \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c) = 0$ (use (6)), hence

$$\frac{\mathrm{Sym}_{\mathbb{F}}^t(\mathrm{St})[i - \frac{k-1}{2}]}{\mathrm{Sym}_{\mathbb{F}}^{t-(q+1)}(\mathrm{St})[i+1 - \frac{k-1}{2}]} \cong \frac{H^0(\widetilde{\mathfrak{X}}, (\mathcal{O}_{\widetilde{\mathfrak{X}}}(k)(i) \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c)}{H^0(\widetilde{\mathfrak{X}}, (\mathcal{O}_{\widetilde{\mathfrak{X}}}(k)(i+1) \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c)}$$

is a representation of $\mathrm{GL}_2(\mathbb{F})$ on the $(q+1)$ -dimensional \mathbb{F} -vector space with basis the \mathbb{F} -rational points of $\mathbb{P}_{\mathbb{F}}^1$. Explicitly, this is the quotient

$$\frac{\mathrm{Sym}_{\mathbb{F}}^t(\mathrm{St})[i - \frac{k-1}{2}]}{\langle X^j Y^{t-j} - X^{q+j-1} Y^{t-q-j+1}; 1 \leq j \leq t-q \rangle_{\mathbb{F}}}.$$

One might ask for its composition series. For example, if $q = 2$, $k = 9$, $i = 0$, $t = 3$, then the class of $X^3 + Y^3 + X^2 Y$ (= the class of $X^3 + Y^3 + XY^2$) in this quotient spans a $\mathrm{GL}_2(\mathbb{F})$ -stable line. The results for even k are similar, with $i > 0$ and $t = \frac{(q-1)k}{2} - i(q+1) \geq q+1$, see also [8]. For the last i , the one for which $q \geq t \geq 0$, we get $\mathrm{Sym}_{\mathbb{F}}^t(\mathrm{St})[i - \frac{k-1}{2}]$ (if k is odd), resp. $\mathrm{Sym}_{\mathbb{F}}^t(\mathrm{St})[i - \frac{k}{2}]$ (if k is even). To complete the picture it remains to observe that for even $k \geq 4$ we have

$$(13) \quad \frac{H^0(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(k))}{H^0(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(k)(1))} \cong \mathrm{Ind}_N^G \mathbf{1}$$

where $N \subset G$ denotes the stabilizer of an (arbitrary) non-oriented edge $\{Z_1, Z_2\} \in F^1$ and $\mathbf{1}$ its trivial representation: use $H^1(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(k)(1)) = 0$.

As an application, if q is odd, Teitelbaum [8] constructs modular forms mod $\hat{\pi}$ of weight $q+1$ (in fact, elements of $H^0(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(q+1)(\frac{q+1}{2} - 1))$) for the entire group $\mathrm{SL}_2(K)$. Here we will do the same if q is even. The action of $\mathrm{SL}_2(K)$ on the set F^0 has two orbits: the orbit F_{even}^0 of $Z_{\gamma_0} \in F^0$ and the orbit F_{odd}^0 of $Z_{\gamma_1} \in F^0$. Choose subsets S_{even} and S_{odd} of $\mathrm{SL}_2(K)$ such that $\gamma \mapsto Z_{\gamma}$ defines bijections $S_{\mathrm{even}} \cong F_{\mathrm{even}}^0$ and $S_{\mathrm{odd}} \cong F_{\mathrm{odd}}^0$. Recall that we fixed a coordinate z on X . For any $\gamma \in \mathrm{SL}_2(K)$ we get another function $z \circ \gamma$ on X .

Theorem 3.4. *The $H^0(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(q+1))$ -elements*

$$b_{q+1}^+ = \sum_{\gamma \in S_{\mathrm{even}}} (\iota_{Z_{\gamma}} \circ \rho_{Z_{\gamma}})((z \circ \gamma^{-1} - (z \circ \gamma^{-1})^q)^{-1})$$

$$b_{q+1}^- = \sum_{\gamma \in S_{\mathrm{odd}}} (\iota_{Z_{\gamma}} \circ \rho_{Z_{\gamma}})((z \circ \gamma^{-1} - (z \circ \gamma^{-1})^q)^{-1})$$

are invariant for $\mathrm{SL}_2(K)$, and interchanged by $\begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$.

□

Now let us look at the modular representations $H^1(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(k))$ for $k < 0$. If k is even we get from

$$0 \longrightarrow \mathcal{O}_{\tilde{\mathfrak{X}}}(k) \longrightarrow \prod_{Z \in F^0} \mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z \longrightarrow \prod_{\{Z_1, Z_2\} \in F^1} \mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_1 \cap Z_2} \longrightarrow 0$$

the exact sequence

$$\begin{aligned} 0 \rightarrow H^0(\tilde{\mathfrak{X}}, \prod_{\{Z_1, Z_2\} \in F^1} \mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_1 \cap Z_2}) \longrightarrow \\ H^1(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(k)) \longrightarrow H^1(\tilde{\mathfrak{X}}, \prod_{Z \in F^0} \mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z) \longrightarrow 0. \end{aligned}$$

Here $H^0(\tilde{\mathfrak{X}}, \prod_{\{Z_1, Z_2\} \in F^1} \mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_1 \cap Z_2})$ is as in (13). If k is odd things are easier because then we have

$$H^1(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(k)) \cong H^1(\tilde{\mathfrak{X}}, \prod_{Z \in F^0} (\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z)^c).$$

Thus for any k , even or odd, we need to understand $H^1(\tilde{\mathfrak{X}}, \prod_{Z \in F^0} (\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z)^c)$ as a G -representation; by (the proof of) 3.1 this means understanding $H^1(\tilde{\mathfrak{X}}, (\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c)$ as a $\mathrm{GL}_2(\mathbb{F})$ -representation. By an explicit computation on $\mathbb{P}_{\mathbb{F}}^1$, using the formulas (4) and (6), we see that Serre duality yields a $\mathrm{GL}_2(\mathbb{F})$ -equivariant isomorphism

$$H^1(\tilde{\mathfrak{X}}, (\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c) \cong \mathrm{Hom}_{\mathbb{F}}(H^0(\tilde{\mathfrak{X}}, (\mathcal{O}_{\tilde{\mathfrak{X}}}(-k+2)(1) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c), \mathbb{F})$$

if k is even, resp.

$$H^1(\tilde{\mathfrak{X}}, (\mathcal{O}_{\tilde{\mathfrak{X}}}(k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c) \cong \mathrm{Hom}_{\mathbb{F}}(H^0(\tilde{\mathfrak{X}}, (\mathcal{O}_{\tilde{\mathfrak{X}}}(-k+2) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c), \mathbb{F})$$

if k is odd. (The duals of these representations have been determined above. For example, for odd $k < 0$, setting $t = \frac{(q-1)(-k-1)}{2} - 1$ we get a canonical G -equivariant isomorphism

$$H^1(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(k)) \cong \mathrm{Ind}_{K \times \mathrm{GL}_2(\mathcal{O}_K)}^G \mathrm{Hom}_{\mathbb{F}}(\mathrm{Sym}_{\mathbb{F}}^t(\mathrm{St})[\frac{k-1}{2}], \mathbb{F}).$$

On the other hand, in section 5 below we will obtain for any $k < 0$, even or odd, G -equivariant isomorphisms

$$H^1(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(k)) \otimes \varepsilon^{-k-1} \cong H^0(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}}(2-k)).$$

4 Harmonic cochains

Fix $k \geq 0$. On $\mathrm{Hom}_{\hat{K}}(\mathrm{Sym}_{\hat{K}}^k(\mathrm{St})[1] \otimes \chi^{-k-2}, \hat{K})$ the G -action is given by $(\gamma.h)(x) = h(\gamma^{-1}.x)$ for $\gamma \in G$, $x \in \mathrm{Sym}_{\hat{K}}^k(\mathrm{St})[1] \otimes \chi^{-k-2}$ and $h \in \mathrm{Hom}_{\hat{K}}(\mathrm{Sym}_{\hat{K}}^k(\mathrm{St})[1] \otimes \chi^{-k-2}, \hat{K})$. — (In everything here and below we could replace $\mathrm{Hom}_{\hat{K}}(\mathrm{Sym}_{\hat{K}}^k(\mathrm{St})[1] \otimes \chi^{-k-2}, \hat{K})$ by the isomorphic G -representation $\mathrm{Sym}_{\hat{K}}^k(\mathrm{St})[-k-1] \otimes \chi^{k+2}$: the isomorphism sends h_j (as defined below) to $X^{k-j}Y^j$.) — We set

$$C^1(k+2) = \prod_{\{Z_1, Z_2\} \in F^1} \mathrm{Hom}_{\hat{K}}(\mathrm{Sym}_{\hat{K}}^k(\mathrm{St})[1] \otimes \chi^{-k-2}, \hat{K}),$$

$$C^0(k+2) = \prod_{Z \in F^0} \text{Hom}_{\widehat{K}}(\text{Sym}_{\widehat{K}}^k(\text{St})[1] \otimes \chi^{-k-2}, \widehat{K})$$

(products of copies of $\text{Hom}_{\widehat{K}}(\text{Sym}_{\widehat{K}}^k(\text{St})[1] \otimes \chi^{-k-2}, \widehat{K})$, indexed by F^1 resp. F^0). On $C^1(k+2)$ we define a G -action by

$$(\gamma \cdot f)_{\{Z_1, Z_2\}} = \gamma(f_{\gamma^{-1}\{Z_1, Z_2\}})$$

for $\gamma \in G$ and $(f_{\{Z_1, Z_2\}})_{\{Z_1, Z_2\}} \in C^1(k+2)$. For $Z \in F^0$ let $\text{sg}(Z) = 1$ if $Z \in F_{\text{even}}^0$ and $\text{sg}(Z) = -1$ if $Z \in F_{\text{odd}}^0$. Moreover let

$$*(Z) = \{Z' \in F^0; \{Z, Z'\} \in F^1\}.$$

Then we have the operator

$$C^1(k+2) \xrightarrow{\Delta} C^0(k+2), \quad (f_{\{Z_1, Z_2\}})_{\{Z_1, Z_2\}} \mapsto (\text{sg}(Z) \sum_{Z' \in *(Z)} f_{\{Z, Z'\}})_Z$$

and we define $C_{\text{har}}^1(k+2)$ by the exact sequence

$$0 \longrightarrow C_{\text{har}}^1(k+2) \longrightarrow C^1(k+2) \xrightarrow{\Delta} C^0(k+2).$$

This is the variant with non-trivial coefficients of the space $C_{\text{har}}^1(\widehat{K})$ of \widehat{K} -valued harmonic cochains on \mathcal{BT} which is defined by the exact sequence

$$(14) \quad 0 \longrightarrow C_{\text{har}}^1(\widehat{K}) \longrightarrow \prod_{\{Z_1, Z_2\} \in F^1} \widehat{K} \xrightarrow{\Delta} \prod_{Z \in F^0} \widehat{K}.$$

Let $\Omega_{\widehat{\mathfrak{X}}}^1$ denote the sheaf of logarithmic differential forms for the morphism of log schemes $\widehat{\mathfrak{X}} \rightarrow \text{Spf}(\mathcal{O}_{\widehat{K}})$ (with log structures defined by the respective special fibres). Define

$$\text{res} : \Gamma(\widehat{\mathfrak{X}}, \Omega_{\widehat{\mathfrak{X}}}^1) \rightarrow C_{\text{har}}^1(\widehat{K})$$

to be the unique G -equivariant morphism of $\mathcal{O}_{\widehat{K}}$ -modules with

$$\text{res}(\eta)_{\{Z_{\gamma_0}, Z_{\gamma_{-1}}\}} = a_{-1}$$

for $\eta \in \Gamma(\widehat{\mathfrak{X}}, \Omega_{\widehat{\mathfrak{X}}}^1)$, where

$$\eta(z) = \sum_{j \in \mathbb{Z}} a_j z^j dz$$

is the Laurent expansion of η on the annulus $]Z_{\gamma_0} \cap Z_{\gamma_{-1}}[= sp^{-1}(Z_{\gamma_0} \cap Z_{\gamma_{-1}}) \subset \widehat{X}$ reducing to $Z_{\gamma_0} \cap Z_{\gamma_{-1}}$. (That $\text{res}(\eta)$ indeed lies in $C_{\text{har}}^1(\widehat{K})$ follows from the residue theorem on \mathbb{P}^1 .) This map also has a version with non-trivial coefficients, as follows. Consider the G -equivariant map

$$\Gamma(\widehat{\mathfrak{X}}, \mathcal{O}_{\widehat{\mathfrak{X}}}(k+2)) \longrightarrow \text{Hom}_{\widehat{K}}(\text{Sym}_{\widehat{K}}^k(\text{St})[1] \otimes \chi^{-k-2}, \Gamma(\widehat{\mathfrak{X}}, \Omega_{\widehat{\mathfrak{X}}}^1)), \quad g \mapsto \Phi_g$$

where Φ_g is defined by

$$\Phi_g(X^i Y^{k-i}) = g(z) z^i dz, \quad 0 \leq i \leq k.$$

We use it to define the G -equivariant map

$$Res^0 : \Gamma(\widehat{\mathfrak{X}}, \mathcal{O}_{\widehat{\mathfrak{X}}}(k+2)) \rightarrow \text{Hom}(\text{Sym}_{\widehat{K}}^k(\text{St})[1] \otimes \chi^{-k-2}, C_{har}^1(\widehat{K})) = C_{har}^1(k+2)$$

$$g \mapsto res \circ \phi_g.$$

We will work with the following more explicit description of Res^0 : it is the unique G -equivariant morphism of $\mathcal{O}_{\widehat{K}}$ -modules with

$$(Res^0(g)_{\{Z_{\gamma_0}, Z_{\gamma_{-1}}\}})(X^i Y^{k-i}) = a_{-i-1}$$

for $g \in \Gamma(\widehat{\mathfrak{X}}, \mathcal{O}_{\widehat{\mathfrak{X}}}(k+2))$ and $0 \leq i \leq k$, where

$$g(z) = \sum_{j \in \mathbb{Z}} a_j z^j$$

is the Laurent expansion of g on the annulus $]Z_{\gamma_0} \cap Z_{\gamma_{-1}}[= sp^{-1}(Z_{\gamma_0} \cap Z_{\gamma_{-1}}) \subset \widehat{X}$ reducing to $Z_{\gamma_0} \cap Z_{\gamma_{-1}}$. Equivalently, $(Res^0(g)_{\{Z_1, Z_2\}})(X^i Y^{k-i})$ for arbitrary $\{Z_1, Z_2\} \in F^1$ can be described follows. Choose a $\gamma \in G$ such that $\gamma \cdot \{Z_1, Z_2\} = \{Z_{\gamma_0}, Z_{\gamma_{-1}}\}$. Let $\sum_{j \in \mathbb{Z}} a_j z^j$ be the Laurent expansion of $\gamma.g$ on $]Z_{\gamma_0} \cap Z_{\gamma_{-1}}[$ and write

$$\gamma.(X^i Y^{k-i}) = \sum_{s=0}^k c_s X^s Y^{k-s}$$

in $\text{Sym}_{\widehat{K}}^k(\text{St})[1] \otimes \chi^{-k-2}$. Then $(Res^0(g)_{\{Z_1, Z_2\}})(X^i Y^{k-i}) = \sum_{s=0}^k a_{-s-1} c_s$. This is independent on the choice of γ .

We want to show that Res^0 is injective and to describe its image. For $Z \in F^0$ choose $\gamma \in G$ with $Z = Z_{\gamma}$ and define

$$L_Z = \gamma. \text{Hom}_{\mathcal{O}_{\widehat{K}}}(\text{Sym}_{\mathcal{O}_{\widehat{K}}}^k(\text{St})[1], \mathcal{O}_{\widehat{K}}) \subset \text{Hom}_{\widehat{K}}(\text{Sym}_{\widehat{K}}^k(\text{St})[1] \otimes \chi^{-k-2}, \widehat{K}).$$

In this definition we consider $\text{Hom}_{\mathcal{O}_{\widehat{K}}}(\text{Sym}_{\mathcal{O}_{\widehat{K}}}^k(\text{St})[1], \mathcal{O}_{\widehat{K}})$ not as a $\text{GL}_2(\mathcal{O}_{\widehat{K}})$ -representation but only as a $\mathcal{O}_{\widehat{K}}$ -submodule of the \widehat{K} -vector space underlying the G -representation $\text{Hom}_{\widehat{K}}(\text{Sym}_{\widehat{K}}^k(\text{St})[1] \otimes \chi^{-k-2}, \widehat{K})$. For $\{Z_1, Z_2\} \in F^1$ we write $L_{\{Z_1, Z_2\}} = L_{Z_1} \cap L_{Z_2}$ and then let

$$Z^1(k+2) = \prod_{\{Z_1, Z_2\} \in F^1} L_{\{Z_1, Z_2\}},$$

$$Z^0(k+2) = \prod_Z \in F^0 L_Z,$$

subspaces of $C^1(k+2)$ resp. of $C^0(k+2)$. We define $Z_{har}^1(k+2)$ by the exact sequence

$$(15) \quad 0 \longrightarrow Z_{har}^1(k+2) \longrightarrow Z^1(k+2) \xrightarrow{\Delta} \prod_{Z \in F^0} L_Z.$$

Lemma 4.1. *The image of Res^0 lies in $Z_{har}^1(k+2)$.*

PROOF: By G -equivariance it suffices to check $Res^0(g)_{\{Z_{\gamma_0}, Z_{\gamma_{-1}}\}} \in L_{Z_{\gamma_0}} \cap L_{Z_{\gamma_{-1}}}$ for all $g \in \Gamma(\widehat{\mathfrak{X}}, \mathcal{O}_{\widehat{\mathfrak{X}}}(k+2))$. Let $g(z) = \sum_{j \in \mathbb{Z}} a_j z^j$ be the Laurent expansion of g on $]Z_{\gamma_0} \cap Z_{\gamma_{-1}}[$. From (2) we deduce

$$(16) \quad \omega(g(P)) \geq 0 \quad \text{for all closed points } P \in]\widetilde{\mathfrak{U}}_{\{Z_{\gamma_0}\}}[$$

$$(17) \quad \omega(g(P)) \geq \frac{-k-2}{2} \quad \text{for all closed points } P \in]\widetilde{\mathfrak{U}}_{\{Z_{\gamma_{-1}}\}}[.$$

From (16) we get $\omega(a_j) \geq 0$ for all j (with a point $P \in]Z_{\gamma_0} \cap Z_{\gamma_{-1}}[$ approach $]\widetilde{\mathfrak{U}}_{\{Z_{\gamma_0}\}}[$), hence $Res^0(g)_{\{Z_{\gamma_0}, Z_{\gamma_{-1}}\}} \in \text{Hom}_{\mathcal{O}_{\widehat{K}}}(\text{Sym}_{\mathcal{O}_{\widehat{K}}}^k(\text{St})[1], \mathcal{O}_{\widehat{K}}) = L_{Z_{\gamma_0}}$. From (17) we get $\omega(a_j) \geq \frac{-(k+2)-2j}{2}$ for all j (with a point $P \in]Z_{\gamma_0} \cap Z_{\gamma_{-1}}[$ approach $]\widetilde{\mathfrak{U}}_{\{Z_{\gamma_{-1}}\}}[$). Now in $\text{Sym}_{\widehat{K}}^k(\text{St})[1] \otimes \chi^{-k-2}$ we have $\gamma_{-1}(X^i Y^{k-i}) = \widehat{\pi}^{k-2i} X^i Y^{k-i}$. Thus $Res^0(g)_{\{Z_{\gamma_0}, Z_{\gamma_{-1}}\}}(\gamma_{-1}(X^i Y^{k-i})) = \widehat{\pi}^{k-2i} Res^0(g)_{\{Z_{\gamma_0}, Z_{\gamma_{-1}}\}}(X^i Y^{k-i}) = \widehat{\pi}^{k-2i} a_{-i-1}$ lies in $\mathcal{O}_{\widehat{K}}$, thus $\gamma_1 \cdot Res^0(g)_{\{Z_{\gamma_0}, Z_{\gamma_{-1}}\}}$ lies in $\text{Hom}_{\mathcal{O}_{\widehat{K}}}(\text{Sym}_{\mathcal{O}_{\widehat{K}}}^k(\text{St})[1], \mathcal{O}_{\widehat{K}})$, thus $Res^0(g)_{\{Z_{\gamma_0}, Z_{\gamma_{-1}}\}}$ lies in $L_{Z_{\gamma_{-1}}}$.

Theorem 4.2.

$$Res^0 : \Gamma(\widehat{\mathfrak{X}}, \mathcal{O}_{\widehat{\mathfrak{X}}}(k+2)) \longrightarrow Z_{har}^1(k+2)$$

is an isomorphism.

PROOF: (i) First we claim that the sequence (15) is also exact on the right. Let $\widetilde{Z}^1(k+2) = Z^1(k+2)/(\widehat{\pi})$ and for $Z \in F^0$ let $\widetilde{L}_Z = L_Z/(\widehat{\pi})$. Then it is enough to show that the map

$$\widetilde{Z}^1(k+2) \xrightarrow{\widetilde{\Delta}} \prod_{Z \in F^0} \widetilde{L}_Z$$

induced by Δ is surjective. For $\{Z_1, Z_2\} \in F^1$ let

$$D_{\{Z_1, Z_2\}}^{Z_1} = \text{Im}(L_{\{Z_1, Z_2\}} \rightarrow \widetilde{L}_{Z_1})$$

$$E_{\{Z_1, Z_2\}} = \text{Im}(L_{\{Z_1, Z_2\}} \rightarrow (L_{Z_1} + L_{Z_2})/(\widehat{\pi}))$$

(images under the natural maps). Note that $\dim_{\mathbb{F}}(D_{\{Z_1, Z_2\}}^{Z_1}) = \frac{k+2}{2}$ and $\dim_{\mathbb{F}}(E_{\{Z_1, Z_2\}}) = 1$ if k is even, and $\dim_{\mathbb{F}}(D_{\{Z_1, Z_2\}}^{Z_1}) = \frac{k+1}{2}$ and $E_{\{Z_1, Z_2\}} = 0$ if k is odd (for explicit descriptions see below). For $Z \in F^0$ let

$$\widetilde{Z}^1(k+2)_Z = \prod_{Z' \in *(Z)} D_{\{Z, Z'\}}^Z.$$

Then $\widetilde{\Delta}$ factors as

$$\widetilde{Z}^1(k+2) \xrightarrow{\beta} \prod_{Z \in F^0} \widetilde{Z}^1(k+2)_Z \xrightarrow{\delta = \prod \delta_Z} \prod_{Z \in F^0} \widetilde{L}_Z$$

where β is the product of the natural projection maps. We have an exact sequence

$$0 \longrightarrow \tilde{Z}^1(k+2) \xrightarrow{\beta} \prod_{Z \in F^0} \tilde{Z}^1(k+2)_Z \xrightarrow{\alpha} \prod_{\{Z_1, Z_2\}} E_{\{Z_1, Z_2\}}$$

where α is defined as

$$\alpha(((g_{Z, Z'})_{Z' \in *(Z)})_{Z \in F^0})_{\{Z_1, Z_2\}} = \text{sg}(Z_1)g_{Z_1, Z_2} + \text{sg}(Z_2)g_{Z_2, Z_1}$$

for $\{Z_1, Z_2\} \in F^1$. For $Z \in F^0$ we define $\tilde{Z}_{har}^1(k+2)_Z$ by the exact sequence

$$0 \longrightarrow \tilde{Z}_{har}^1(k+2)_Z \xrightarrow{\nu_Z} \tilde{Z}^1(k+2)_Z \xrightarrow{\delta_Z} \tilde{L}_Z.$$

Now it is enough to prove that each δ_Z (and hence δ) is surjective, and that

$$\prod_{Z \in F^0} \tilde{Z}_{har}^1(k+2)_Z \xrightarrow{\alpha \circ (\prod_Z \nu_Z)} \prod_{\{Z_1, Z_2\}} E_{\{Z_1, Z_2\}}$$

is surjective. The surjectivity of $\alpha \circ (\prod_Z \nu_Z)$, an empty statement if k is odd, will be implied by the surjectivity of its factors

$$\tilde{Z}_{har}^1(k+2)_{Z_1} \xrightarrow{\mu_{Z_1, Z_2}} E_{\{Z_1, Z_2\}}.$$

Let us make the objects explicit. By equivariance we may assume $Z = Z_{\gamma_0}$, resp. $\{Z_1, Z_2\} = \{Z_{\gamma_0}, Z_{\gamma_{-1}}\}$. For $0 \leq j \leq k$ define $h_j \in \text{Hom}_{\hat{K}}(\text{Sym}_{\hat{K}}^k(\text{St})[1] \otimes \chi^{-k-2}, \hat{K})$ by

$$h_j(X^i Y^{k-i}) = \begin{cases} 1 & : & i = j \\ 0 & : & i \neq j \end{cases}.$$

Then one finds

$$L_{\gamma_0} = \oplus_{j=0}^k \mathcal{O}_{\hat{K}}.h_j, \quad L_{\gamma_1} = \oplus_{j=0}^k (\hat{\pi}^{k-2j}).h_j, \quad L_{\gamma_{-1}} = \oplus_{j=0}^k (\hat{\pi}^{2j-k}).h_j,$$

$$D_{\{Z_{\gamma_{-1}}, Z_{\gamma_0}\}}^{Z_{\gamma_0}} = \oplus_{j=0}^{\lfloor \frac{k}{2} \rfloor} \mathbb{F}.h_j, \quad D_{\{Z_{\gamma_1}, Z_{\gamma_0}\}}^{Z_{\gamma_0}} = \oplus_{j=\lceil \frac{k}{2} \rceil}^k \mathbb{F}.h_j$$

and if k is even also $E_{\{Z_{\gamma_{-1}}, Z_{\gamma_0}\}} = \mathbb{F}.h_{\frac{k}{2}}$. The surjectivity of $\delta_{Z_{\gamma_0}}$ follows from $\tilde{L}_{Z_{\gamma_0}} = D_{\{Z_{\gamma_{-1}}, Z_{\gamma_0}\}}^{Z_{\gamma_0}} + D_{\{Z_{\gamma_1}, Z_{\gamma_0}\}}^{Z_{\gamma_0}}$. For the surjectivity of $\mu_{Z_{\gamma_0}, Z_{\gamma_{-1}}}$ (if k is even): the element $h_{\frac{k}{2}} \in E_{\{Z_{\gamma_{-1}}, Z_{\gamma_0}\}}$ is the image of the $\tilde{Z}_{har}^1(k+2)_{Z_{\gamma_0}}$ -element with entry $h_{\frac{k}{2}}$ in the $\{Z_{\gamma_{-1}}, Z_{\gamma_0}\}$ -component, with entry $-h_{\frac{k}{2}}$ in the $\{Z_{\gamma_1}, Z_{\gamma_0}\}$ -component, and with entry 0 at all other components.

(ii) Let $\tilde{Z}_{har}^1(k+2) = Z_{har}^1(k+2)/(\hat{\pi})$. To prove the theorem, since $\Gamma(\hat{\mathfrak{X}}, \mathcal{O}_{\hat{\mathfrak{X}}}(k+2))$ and $Z_{har}^1(k+2)$ are $\hat{\pi}$ -adically complete and separated, and since $Z_{har}^1(k+2)$ is $\mathcal{O}_{\hat{K}}$ -flat, it is enough to prove that the induced map

$$\widetilde{\text{Res}}^0 : \Gamma(\hat{\mathfrak{X}}, \mathcal{O}_{\hat{\mathfrak{X}}}(k+2))/(\hat{\pi}) \rightarrow \tilde{Z}_{har}^1(k+2)$$

is an isomorphism. Since $\prod_{Z \in F^0} L_Z$ is $\mathcal{O}_{\widehat{K}}$ -flat it follows from (i) that (15) reduces modulo $(\widehat{\pi})$ to an exact sequence

$$0 \longrightarrow \widetilde{Z}_{har}^1(k+2) \longrightarrow \widetilde{Z}^1(k+2) \xrightarrow{\widetilde{\Delta}} \prod_{Z \in F^0} \widetilde{L}_Z.$$

We then also obtain from (i) for any $Z \in F^0$ exact sequences

$$0 \rightarrow \widetilde{Z}_{har}^1(k+2) \rightarrow \prod_{Z \in F^0} \widetilde{Z}_{har}^1(k+2)_Z \rightarrow \prod_{\{Z_1, Z_2\} \in F^1} E_{\{Z_1, Z_2\}} \rightarrow 0$$

and (by surjectivity of δ_Z) the estimates

$$\dim_{\mathbb{F}}(\widetilde{Z}_{har}^1(k+2)_Z) = \begin{cases} \frac{(q-1)(k+1)}{2} & : k \text{ odd} \\ \frac{(q-1)(k+2)}{2} + 1 & : k \text{ even} \end{cases}$$

Now let us look at the source of \widetilde{Res}^0 . By 2.1 we know that this is $H^0(\mathfrak{X}, \mathcal{O}_{\widetilde{\mathfrak{X}}}(k+2))$. Our discussion in section 2 implies that the natural restriction maps induce an exact sequence (note $H^1(\mathfrak{X}, \mathcal{O}_{\widetilde{\mathfrak{X}}}(k+2)) = 0$)

$$0 \rightarrow H^0(\mathfrak{X}, \mathcal{O}_{\widetilde{\mathfrak{X}}}(k+2)) \rightarrow \prod_{Z \in F^0} H^0(\mathfrak{X}, (\mathcal{O}_{\widetilde{\mathfrak{X}}}(k+2) \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} \mathcal{O}_Z)^c) \rightarrow \prod_{\{Z_1, Z_2\}} J_{\{Z_1, Z_2\}} \rightarrow 0$$

where $\dim_{\mathbb{F}}(J_{\{Z_1, Z_2\}}) = 1$ if k is even, and $J_{\{Z_1, Z_2\}} = 0$ if k is odd. Since \widetilde{Res}^0 induces isomorphisms $J_{\{Z_1, Z_2\}} \cong E_{\{Z_1, Z_2\}}$ it now suffices to see that the map

$$H^0(\mathfrak{X}, (\mathcal{O}_{\widetilde{\mathfrak{X}}}(k+2) \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} \mathcal{O}_Z)^c) \rightarrow \widetilde{Z}_{har}^1(k+2)_Z$$

induced by \widetilde{Res}^0 is an isomorphism for any $Z \in F^0$, or, by equivariance, for $Z = Z_{\gamma_0}$. Recall that identifying $\mathbb{P}_{\mathbb{F}}^1 \cong Z_{\gamma_0}$ as before we have

$$(\mathcal{O}_{\widetilde{\mathfrak{X}}}(k+2) \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c \cong \begin{cases} \mathcal{L}(\frac{-k-3}{2} \cdot \infty + \sum_{b \in \mathbb{F}} \frac{k+1}{2} \cdot b) & : k \text{ odd} \\ \mathcal{L}(\frac{-k-2}{2} \cdot \infty + \sum_{b \in \mathbb{F}} \frac{k+2}{2} \cdot b) & : k \text{ even} \end{cases}.$$

For k odd, if $g \in H^0(\mathfrak{X}, (\mathcal{O}_{\widetilde{\mathfrak{X}}}(k+2) \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c) = H^0(\mathbb{P}_{\mathbb{F}}^1, \mathcal{L}(\frac{-k-3}{2} \cdot \infty + \sum_{b \in \mathbb{F}} \frac{k+1}{2} \cdot b))$ lies in the kernel of \widetilde{Res}^0 then it is an element even of $H^0(\mathbb{P}_{\mathbb{F}}^1, \mathcal{L}(\frac{-k-3}{2} \cdot \infty))$ and therefore it vanishes. Thus \widetilde{Res}^0 is injective. Similarly for even k . On the other hand by our above computation we find $\dim_{\mathbb{F}}(H^0(\mathfrak{X}, (\mathcal{O}_{\widetilde{\mathfrak{X}}}(k+2) \otimes_{\mathcal{O}_{\widetilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c)) = \dim_{\mathbb{F}}(\widetilde{Z}_{har}^1(k+2)_Z)$, thus \widetilde{Res}^0 is also surjective and the proof is complete. \square

The p -adic Shimura isomorphism [6] p.98 is an immediate consequence of 4.2.

5 The reduced de Rham complex

In this section $\text{char}(K) = 0$. Fix $k \geq 0$ and for our fixed coordinate z let $\partial = \frac{d}{dz}$. Let $(\Omega_X^\bullet \otimes_K \text{Sym}_K^k(\text{St}), \partial \otimes \text{id})$ be the de Rham complex on X with coefficients in $\text{Sym}_K^k(\text{St})$. By [6] p.97 this complex is $\text{SL}_2(K)$ -equivariantly quasi-isomorphic with the "reduced de Rham complex"

$$\mathcal{R}_X^\bullet = [\mathcal{O}_X(-k) \xrightarrow{\partial^{k+1}} \mathcal{O}_X(k+2)]$$

on X . (The genesis of this "theta operator" ∂^{k+1} from $(\Omega_X^\bullet \otimes_K \text{Sym}_K^k(\text{St}), \partial \otimes \text{id})$ is completely parallel to that of the theta operator on classical modular forms, cf. [2]).

We change bases $K \rightarrow \widehat{K}$. Since $\omega(z(P)) = -n$ for any n and any point $P \in (\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}})_{\widehat{K}}$ the operator ∂ on $\mathcal{O}_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}}} \otimes_{\widehat{K}} \widehat{K} = sp_* \mathcal{O}_{(\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}})_{\widehat{K}}}$ restricts to a map $\partial : \mathcal{O}_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}}} \rightarrow \pi^n \cdot \mathcal{O}_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}}}$. Iterating we get a map $\partial^{k+1} : \widehat{\pi}^{-kn} \cdot \mathcal{O}_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}}} \rightarrow \widehat{\pi}^{(k+2)n} \cdot \mathcal{O}_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}}}$, i.e. a map $\partial^{k+1} : \mathcal{O}_{\widehat{\mathfrak{X}}}(-k)|_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}}} \rightarrow \mathcal{O}_{\widehat{\mathfrak{X}}}(k+2)|_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}}}$. By equivariance we see that ∂^{k+1} induces a map $\partial^{k+1} : \mathcal{O}_{\widehat{\mathfrak{X}}}(-k)|_{\widehat{\mathfrak{U}}_{\{Z\}}} \rightarrow \mathcal{O}_{\widehat{\mathfrak{X}}}(k+2)|_{\widehat{\mathfrak{U}}_{\{Z\}}}$ for any $Z \in F^0$. By an argument similar to that at the end of the proof of 1.1 it follows that ∂^{k+1} respects these integral structures also above the singular points of $\widetilde{\mathfrak{X}}$, hence a complex

$$\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet = [\mathcal{O}_{\widehat{\mathfrak{X}}}(-k) \xrightarrow{\partial^{k+1}} \mathcal{O}_{\widehat{\mathfrak{X}}}(k+2)].$$

We denote by $\mathcal{H}^i(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet)$ for $i = 0$ and $i = 1$ the cohomology sheaves.

Theorem 5.1. *For any i, j we have canonical isomorphisms*

$$H^j(\widetilde{\mathfrak{X}}, \mathcal{H}^i(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet)) \cong H^j(\widetilde{\mathfrak{X}}, \mathcal{R}_{\widehat{\mathfrak{X}}}^i).$$

PROOF: For $i = 0$ the map is induced by the canonical injection $\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet) \rightarrow \mathcal{R}_{\widehat{\mathfrak{X}}}^0 = \mathcal{O}_{\widehat{\mathfrak{X}}}(-k)$, for $i = 1$ it is induced by the canonical surjection $\mathcal{R}_{\widehat{\mathfrak{X}}}^1 \rightarrow \mathcal{H}^1(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet)$. Once we know the claim for $i = 0$ it follows that $H^*(\widetilde{\mathfrak{X}}, \mathcal{B}) = 0$ for $\mathcal{B} = \text{Im}(\mathcal{R}_{\widehat{\mathfrak{X}}}^0 \rightarrow \mathcal{R}_{\widehat{\mathfrak{X}}}^1) = \text{Ker}(\mathcal{R}_{\widehat{\mathfrak{X}}}^1 \rightarrow \mathcal{H}^1(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet))$, hence the claim for $i = 1$. Thus we concentrate on the case $i = 0$. Denote by $(\cdot)_m$ reduction modulo $\widehat{\pi}^m$. Since $\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet) = \lim_{\leftarrow m} (\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet))_m$ and $\mathcal{R}_{\widehat{\mathfrak{X}}}^0 = \lim_{\leftarrow m} (\mathcal{R}_{\widehat{\mathfrak{X}}}^0)_m$, the spectral sequence for the composition of derived functors $\mathbb{R} \lim_{\leftarrow m} \mathbb{R} \Gamma(\widetilde{\mathfrak{X}}, \cdot)$ shows that it suffices to show

$$H^j(\widetilde{\mathfrak{X}}, (\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet))_m) \cong H^j(\widetilde{\mathfrak{X}}, (\mathcal{R}_{\widehat{\mathfrak{X}}}^0)_m)$$

for any m . Now $\mathcal{R}_{\widehat{\mathfrak{X}}}^0$ and hence also its subsheaf $\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet)$ is $\mathcal{O}_{\widehat{K}}$ -flat. Therefore one gets exact sequences of sheaves

$$0 \rightarrow \mathcal{F}_{m-1} \xrightarrow{\widehat{\pi}^{m-1}} \mathcal{F}_m \rightarrow \mathcal{F}_1 \rightarrow 0$$

for $\mathcal{F} = \mathcal{R}_{\widehat{\mathfrak{X}}}^0$ and $\mathcal{F} = \mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet)$. Using the associated long exact cohomology sequences we reduce our task to proving the isomorphism just stated in the case $m = 1$. Now observe

that $\mathcal{H}^0(\mathcal{R}_X^\bullet \otimes_K \widehat{K})$ is precisely the locally constant sheaf generated by the \widehat{K} -vector space of polynomials in the variable z of degree at most k . Thus $\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet)$ consists of such polynomials subject to growth conditions. Namely, since $\mathcal{R}_{\widehat{\mathfrak{X}}}^0|_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}}} = \mathcal{O}_{\widehat{\mathfrak{X}}}(-k)|_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}}} = \widehat{\pi}^{-kn} \mathcal{O}_{\widehat{\mathfrak{X}}}|_{\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}}}$ and $\omega(z(P)) = -n$ for any n and any point $P \in (\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}})_{\widehat{K}}$ we have

$$\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet)(\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}}) = \left\{ \sum_{0 \leq t \leq k} d_t z^t \mid d_t \in \widehat{K}, \omega(d_t) \geq tn - \frac{kn}{2} \right\},$$

$$\begin{aligned} \mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet)(\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n-1}}\}}) &= \mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet)(\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}}) \cap \mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet)(\widehat{\mathfrak{U}}_{\{Z_{\gamma_{n-1}}\}}) \\ &= \left\{ \sum_{0 \leq t \leq k} d_t z^t \mid d_t \in \widehat{K}, \omega(d_t) \geq \begin{cases} tn - \frac{kn}{2} & : t \geq \frac{k}{2} \\ t(n-1) - \frac{k(n-1)}{2} & : t \leq \frac{k}{2} \end{cases} \right\} \end{aligned}$$

(any k , even or odd). For $Z \in F^0$ let $(\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet))_1^Z$ be the image of the composition

$$\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet) \rightarrow \mathcal{R}_{\widehat{\mathfrak{X}}}^0 = \mathcal{O}_{\widehat{\mathfrak{X}}}(-k) \rightarrow (\mathcal{O}_{\widehat{\mathfrak{X}}}(-k) \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_Z)^c.$$

Then the above shows

$$\begin{aligned} (\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet))_1^{Z_{\gamma_n}}(\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}\}}) &= \left\{ \sum_{0 \leq t \leq k} \bar{d}_t z^t \mid \bar{d}_t \in \frac{(\widehat{\pi}^{(2t-k)n})}{(\widehat{\pi}^{(2t-k)n+1})} \right\} \\ (\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet))_1^{Z_{\gamma_n}}(\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n-1}}\}}) &= \left\{ \sum_{\frac{k}{2} \leq t \leq k} \bar{d}_t z^t \mid \bar{d}_t \in \frac{(\widehat{\pi}^{(2t-k)n})}{(\widehat{\pi}^{(2t-k)n+1})} \right\} \\ (\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet))_1^{Z_{\gamma_n}}(\widehat{\mathfrak{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}) &= \left\{ \sum_{0 \leq t \leq \frac{k}{2}} \bar{d}_t z^t \mid \bar{d}_t \in \frac{(\widehat{\pi}^{(2t-k)n})}{(\widehat{\pi}^{(2t-k)n+1})} \right\}. \end{aligned}$$

Similar descriptions hold at other $Z \in F^0$, resp. $\{Z_1, Z_2\} \in F^1$, by equivariance. We find

$$(\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet))_1 = \prod_{Z \in F^0} (\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet))_1^Z$$

if k is odd (because then there are no summands $\bar{d}_{\frac{k}{2}} z^{\frac{k}{2}}$ to consider). If k is even we find an exact sequence

$$0 \rightarrow (\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet))_1 \rightarrow \prod_{Z \in F^0} (\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet))_1^Z \rightarrow \prod_{\{Z_1, Z_2\} \in F^1} (\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet))_1^{Z_1, Z_2} \rightarrow 0$$

where $(\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet))_1^{Z_1, Z_2}$ for $\{Z_1, Z_2\} \in F^1$ is a sheaf with $(\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet))_1^{Z_1, Z_2}(U) \cong \mathbb{F}$ if $U \cap Z_1 \cap Z_2 \neq \emptyset$, and $= 0$ for other open $U \subset \widehat{\mathfrak{X}}$. On the other hand we have

$$(\mathcal{R}_{\widehat{\mathfrak{X}}}^0)_1 = \mathcal{O}_{\widehat{\mathfrak{X}}}(-k) = \prod_{Z \in F^0} (\mathcal{O}_{\widehat{\mathfrak{X}}}(-k) \otimes_{\mathcal{O}_{\widehat{\mathfrak{X}}}} \mathcal{O}_Z)^c$$

if k is odd, and an exact sequence

$$0 \rightarrow (\mathcal{R}_{\tilde{\mathfrak{X}}}^0)_1 \rightarrow \prod_{Z \in F^0} (\mathcal{O}_{\tilde{\mathfrak{X}}}(-k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z)^c \rightarrow \prod_{\{Z_1, Z_2\} \in F^1} \mathcal{O}_{\tilde{\mathfrak{X}}}(-k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_1 \cap Z_2} \rightarrow 0$$

if k is even. For $Z \in F^0$ let

$$\alpha_Z : (\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^Z \rightarrow (\mathcal{O}_{\tilde{\mathfrak{X}}}(-k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_Z)^c$$

be the inclusion. If k is even then for $\{Z_1, Z_2\} \in F^1$ the maps α_{Z_1} and α_{Z_2} commute with obvious isomorphisms

$$\alpha_{Z_1, Z_2} : (\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^{Z_1, Z_2} \rightarrow \mathcal{O}_{\tilde{\mathfrak{X}}}(-k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_1 \cap Z_2}.$$

Since the α_Z also commute with our map $(\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1 \rightarrow (\mathcal{R}_{\tilde{\mathfrak{X}}}^0)_1$ in question, it remains to prove that the α_Z induce isomorphisms in cohomology. By equivariance it is enough to do this for $Z = Z_{\gamma_0}$. We identify $\text{Spec}(\mathbb{F}[z]) \cup \{\infty\} = \mathbb{P}_{\mathbb{F}}^1 \cong Z_{\gamma_0}$ such that this z on $\mathbb{P}_{\mathbb{F}}^1$ is induced by the global variable z on X . In particular, $\infty \in \mathbb{P}_{\mathbb{F}}^1$ corresponds to $Z_{\gamma_0} \cap Z_{\gamma_1}$, and $0 \in \mathbb{P}_{\mathbb{F}}^1$ corresponds to $Z_{\gamma_0} \cap Z_{\gamma_{-1}}$. Let $\iota : \mathbb{P}_{\mathbb{F}}^1 \cong Z_{\gamma_0} \rightarrow \tilde{\mathfrak{X}}$ be the closed immersion. Since we have

$$H^*(\tilde{\mathfrak{X}}, \mathcal{F}) = H^*(\mathbb{P}_{\mathbb{F}}^1, \iota^{-1}\mathcal{F})$$

for both $\mathcal{F} = (\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^{Z_{\gamma_0}}$ and $\mathcal{F} = (\mathcal{O}_{\tilde{\mathfrak{X}}}(-k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c$, we must show that

$$H^*(\mathbb{P}_{\mathbb{F}}^1, \iota^{-1}(\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^{Z_{\gamma_0}}) \rightarrow H^*(\mathbb{P}_{\mathbb{F}}^1, \iota^{-1}(\mathcal{O}_{\tilde{\mathfrak{X}}}(-k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c)$$

is an isomorphism. If on $\mathbb{P}_{\mathbb{F}}^1$ we define the divisor

$$D = \begin{cases} \frac{k}{2} \cdot \infty - \sum_{b \in \mathbb{F}} \frac{k}{2} \cdot b & : k \text{ even} \\ \frac{k+1}{2} \cdot \infty - \sum_{b \in \mathbb{F}} \frac{k-1}{2} \cdot b & : k \text{ odd} \end{cases}$$

then we have a natural identification

$$\mathcal{L}(D) = \iota^{-1}(\mathcal{O}_{\tilde{\mathfrak{X}}}(-k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c.$$

In this way we may view $\iota^{-1}(\mathcal{O}_{\tilde{\mathfrak{X}}}(-k) \otimes_{\mathcal{O}_{\tilde{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c$ as a subsheaf of $\mathcal{L}(k \cdot \infty)$. On the other hand we may view $\iota^{-1}(\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^{Z_{\gamma_0}}$ as a subsheaf of the constant \mathbb{F} -vector space sheaf \mathcal{H} on $\mathbb{P}_{\mathbb{F}}^1$ with value $\oplus_{i=0}^k \mathbb{F} \cdot z^i$ (as a sub \mathbb{F} -vector space of the function field $\mathbb{F}(z)$). The inclusion $\beta : \mathcal{H} \rightarrow \mathcal{L}(k \cdot \infty)$ induces our map $\iota^{-1}(\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^{Z_{\gamma_0}} \rightarrow \mathcal{L}(D)$ in question. It also induces an isomorphism between the respective cokernel (skyscraper) sheaves

$$\frac{\mathcal{H}}{\iota^{-1}(\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^{Z_{\gamma_0}}} \cong \frac{\mathcal{L}(k \cdot \infty)}{\mathcal{L}(D)}$$

(use the above local description of $\iota^{-1}(\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^{Z_{\gamma_0}}$). Since clearly β induces isomorphisms

$$H^*(\mathbb{P}_{\mathbb{F}}^1, \mathcal{H}) \cong H^*(\mathbb{P}_{\mathbb{F}}^1, \mathcal{L}(k \cdot \infty))$$

we are done. □

Corollary 5.2. *We have the Hodge decomposition*

$$(18) \quad H^1(\tilde{\mathfrak{X}}, \mathcal{R}_{\hat{\mathfrak{X}}}^\bullet) \cong H^0(\tilde{\mathfrak{X}}, \mathcal{O}_{\hat{\mathfrak{X}}}(k+2)) \oplus H^1(\tilde{\mathfrak{X}}, \mathcal{O}_{\hat{\mathfrak{X}}}(-k)).$$

PROOF: Consider the canonical maps of sheaf complexes

$$\begin{aligned} [\mathcal{H}^0(\mathcal{R}_{\hat{\mathfrak{X}}}^\bullet) \xrightarrow{0} \mathcal{R}_{\hat{\mathfrak{X}}}^1] &\longrightarrow \mathcal{R}_{\hat{\mathfrak{X}}}^\bullet \\ [\mathcal{H}^0(\mathcal{R}_{\hat{\mathfrak{X}}}^\bullet) \xrightarrow{0} \mathcal{R}_{\hat{\mathfrak{X}}}^1] &\longrightarrow [\mathcal{R}_{\hat{\mathfrak{X}}}^0 \xrightarrow{0} \mathcal{R}_{\hat{\mathfrak{X}}}^1] \end{aligned}$$

on $\hat{\mathfrak{X}}$. By 5.1 both of them induce isomorphisms in cohomology; together we thus obtain the isomorphism

$$\mathbb{R}\Gamma(\tilde{\mathfrak{X}}, \mathcal{R}_{\hat{\mathfrak{X}}}^\bullet) \cong \mathbb{R}\Gamma(\tilde{\mathfrak{X}}, [\mathcal{R}_{\hat{\mathfrak{X}}}^0 \xrightarrow{0} \mathcal{R}_{\hat{\mathfrak{X}}}^1]).$$

We derive the stated Hodge decomposition. \square

Let again $\Gamma < \mathrm{SL}_2(K)$ be a cocompact discrete torsion free subgroup.

Theorem 5.3. (a) *The reduced Hodge spectral sequence*

$$E_1^{r,s} = H^s(X_\Gamma, (\mathcal{R}_X^r)^\Gamma) \Rightarrow H^{r+s}(X_\Gamma, (\mathcal{R}_X^\bullet)^\Gamma) = H^{r+s}(X_\Gamma, (\Omega_X^\bullet \otimes_K \mathrm{Sym}_K^k(\mathrm{St}))^\Gamma)$$

degenerates in E_1 .

(b) $H^1(X_\Gamma, (\Omega_X^\bullet \otimes_K \mathrm{Sym}_K^k(\mathrm{St}))^\Gamma) = H^1(X_\Gamma, (\mathcal{R}_X^\bullet)^\Gamma)$ decomposes naturally as

$$H^1(X_\Gamma, (\Omega_X^\bullet \otimes_K \mathrm{Sym}_K^k(\mathrm{St}))^\Gamma) = H^1(\Gamma, \mathrm{Sym}_K^k(\mathrm{St})) \oplus H^0(X_\Gamma, \mathcal{O}_X(k+2)^\Gamma).$$

(c) *If Γ is the free group on g generators and if $k > 0$, then*

$$\dim_K(H^1(X_\Gamma, \mathcal{O}_X(-k)^\Gamma)) = \dim_K(H^0(X_\Gamma, \mathcal{O}_X(k+2)^\Gamma)) = (g-1)(k+1).$$

PROOF: We may of course change bases from K to \hat{K} . Statement (a) is a consequence of 2.2 (if $k > 0$) but we can also argue as follows. It is enough to show that the inclusion of sheaf complexes

$$[\mathcal{H}^0(\mathcal{R}_{\hat{\mathfrak{X}}}^\bullet)^\Gamma \xrightarrow{0} (\mathcal{R}_{\hat{\mathfrak{X}}}^1)^\Gamma] \hookrightarrow (\mathcal{R}_{\hat{\mathfrak{X}}}^\bullet)^\Gamma$$

on \hat{X}_Γ induces isomorphisms

$$H^*(\tilde{X}_\Gamma, [\mathcal{H}^0(\mathcal{R}_{\hat{\mathfrak{X}}}^\bullet)^\Gamma \xrightarrow{0} (\mathcal{R}_{\hat{\mathfrak{X}}}^1)^\Gamma]) \cong H^*(\tilde{X}_\Gamma, (\mathcal{R}_{\hat{\mathfrak{X}}}^\bullet)^\Gamma).$$

For this it suffices to show that $\mathcal{H}^0(\mathcal{R}_{\hat{\mathfrak{X}}}^\bullet)^\Gamma \rightarrow (\mathcal{R}_{\hat{\mathfrak{X}}}^0)^\Gamma$ induces isomorphisms in cohomology. Now \hat{X}_Γ is quasi-compact, hence $H^*(\tilde{X}_\Gamma, \cdot)$ commutes with $(\cdot) \otimes_{\mathcal{O}_{\hat{K}}} \hat{K}$. Therefore it suffices to show that the morphism of sheaves $\mathcal{H}^0(\mathcal{R}_{\hat{\mathfrak{X}}}^\bullet)^\Gamma \rightarrow (\mathcal{R}_{\hat{\mathfrak{X}}}^0)^\Gamma$ on $\hat{\mathfrak{X}}_\Gamma$ induces isomorphisms

$$(19) \quad H^*(\tilde{\mathfrak{X}}_\Gamma, \mathcal{H}^0(\mathcal{R}_{\hat{\mathfrak{X}}}^\bullet)^\Gamma) \cong H^*(\tilde{\mathfrak{X}}_\Gamma, (\mathcal{R}_{\hat{\mathfrak{X}}}^0)^\Gamma).$$

Using the covering spectral sequences

$$E_2^{r,s} = H^r(\Gamma, H^s(\widehat{\mathfrak{X}}, \mathcal{F})) \Rightarrow H^{r+s}(\widehat{\mathfrak{X}}_\Gamma, \mathcal{F}^\Gamma)$$

for $\mathcal{F} = \mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet)$ and $\mathcal{F} = \mathcal{R}_{\widehat{\mathfrak{X}}}^0$ we see that it is enough to prove that the maps

$$H^r(\Gamma, H^s(\widehat{\mathfrak{X}}, \mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet))) \rightarrow H^r(\Gamma, H^s(\widehat{\mathfrak{X}}, \mathcal{R}_{\widehat{\mathfrak{X}}}^0))$$

are isomorphisms. But they are, as follows from 5.1. We turn to (b). We have

$$\begin{aligned} H^1(\widehat{X}_\Gamma, (\mathcal{R}_{\widehat{X}}^\bullet)^\Gamma) &= H^1(\widehat{\mathfrak{X}}_\Gamma, (\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet)^\Gamma) \otimes_{\mathcal{O}_{\widehat{K}}} \widehat{K} \\ &= H^1(\widehat{\mathfrak{X}}_\Gamma, \mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet)^\Gamma) \otimes_{\mathcal{O}_{\widehat{K}}} \widehat{K} \oplus H^0(\widehat{\mathfrak{X}}_\Gamma, (\mathcal{R}_{\widehat{\mathfrak{X}}}^1)^\Gamma) \otimes_{\mathcal{O}_{\widehat{K}}} \widehat{K} \\ &= H^1(\widehat{X}_\Gamma, \mathcal{H}^0(\mathcal{R}_{\widehat{X}}^\bullet)^\Gamma) \oplus H^0(\widehat{X}_\Gamma, (\mathcal{R}_{\widehat{X}}^1)^\Gamma) \end{aligned}$$

where the first and the third equality follow again from the quasi-compactness of \widehat{X}_Γ , and the second equality from (19). Now $\mathcal{R}_X^1 = \mathcal{O}_X(k+2)$, and on the other hand

$$H^1(X_\Gamma, \mathcal{H}^0(\mathcal{R}_X^\bullet)^\Gamma) = H^1(\Gamma, \mathbb{R}\Gamma(X, \mathcal{H}^0(\mathcal{R}_X^\bullet))).$$

But $H^0(X, \mathcal{H}^0(\mathcal{R}_X^\bullet)) = H^0(X, \mathcal{R}_X^\bullet) = \text{Sym}_K^k(\text{St})$ and $H^j(X, \mathcal{H}^0(\mathcal{R}_X^\bullet)) = 0$ for $j \neq 0$ because $\mathcal{H}^0(\mathcal{R}_X^\bullet)$ is the locally constant sheaf on X generated by $H^0(X, \mathcal{R}_X^\bullet) = H^0(X, \Omega_X^\bullet \otimes_K \text{Sym}_K^k(\text{St})) = \text{Sym}_K^k(\text{St})$. In (c) for the equality $\dim_K(H^0(X_\Gamma, \mathcal{O}_X(k+2)^\Gamma)) = (g-1)(k+1)$ see [6] p.98. The equality $\dim_K(H^1(X_\Gamma, \mathcal{O}_X(-k)^\Gamma)) = (g-1)(k+1)$ follows from statement (a) together with [5] p.628 and [6] p.98. \square

The decomposition in (b) is not new. It was established for the first time in [7] and later again in [5]. Both these (mutually different) proofs use sophisticated analytic methods (e.g. Coleman integration in [7]). The degeneration of the spectral sequence in (a) however, conjectured in [5], seemed to be unknown before (cf. [5] p.649). Note that the spectral sequence for the *non-reduced* de Rham complex does not degenerate in general at E_1 (cf. loc. cit.).

Corollary 5.4. *The intersection of $H^0(\widehat{\mathfrak{X}}, \mathcal{O}_{\widehat{\mathfrak{X}}}(k+2))$ and of*

$$\text{Im}[H^0(\widetilde{X}, \mathcal{O}_{\widehat{X}}(-k)) \xrightarrow{\partial^{k+1}} H^0(\widetilde{X}, \mathcal{O}_{\widehat{X}}(k+2))]$$

inside $H^0(\widehat{\mathfrak{X}}, sp_ \mathcal{O}_{\widehat{X}}(k+2)) = H^0(\widetilde{X}, \mathcal{O}_{\widehat{X}}(k+2))$ is zero. In particular, $H^0(\widehat{\mathfrak{X}}, \mathcal{O}_{\widehat{\mathfrak{X}}}(k+2))$ can be viewed as a submodule of $H^1(\widetilde{X}, \mathcal{R}_{\widehat{X}}^\bullet) = H^1(\widetilde{X}, \Omega_{\widehat{X}}^\bullet \otimes_K \text{Sym}_K^k(\text{St}))$.*

PROOF: This follows immediately from the injectivity of the map Res^0 in 4.2. \square

Theorem 5.5. *For $k > 0$ there is a natural G -equivariant isomorphism*

$$\theta : Z_{\text{har}}^1(k+2) \otimes \varepsilon^{k+1} \cong H^1(\widehat{\mathfrak{X}}, \mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet)).$$

PROOF: Observe $\varepsilon = \det \cdot \chi^{-2}$, which relates the twisting here to that in Section 4. We have a G -equivariant isomorphism

$$\mathrm{Hom}_{\widehat{K}}(\mathrm{Sym}_{\widehat{K}}^k(\mathrm{St})[-k] \otimes \chi^k, \widehat{K}) \xrightarrow{\sigma} H^0(\widetilde{X}, \mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet)), \quad h_j \mapsto z^{k-j}$$

with $h_j \in \mathrm{Hom}_{\widehat{K}}(\mathrm{Sym}_{\widehat{K}}^k(\mathrm{St})[-k] \otimes \chi^k, \widehat{K})$ as in the proof of 4.2, i.e. $h_j(X^j Y^{k-j}) = 1$ and $h_j(X^i Y^{k-i}) = 0$ for $i \neq j$. For $Z \in F^0$ and $\{Z_1, Z_2\} \in F^1$ we define sheaves \mathcal{G}_Z and $\mathcal{G}_{\{Z_1, Z_2\}}$ on $\widehat{\mathfrak{X}}$: for open $U \subset \widehat{\mathfrak{X}}$ we let

$$\mathcal{G}_Z(U) = \begin{cases} \mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet)(\widehat{\mathfrak{U}}_{\{Z\}}) & : U \cap Z \neq \emptyset \\ 0 & : U \cap Z = \emptyset \end{cases}$$

$$\mathcal{G}_{\{Z_1, Z_2\}}(U) = \begin{cases} \mathcal{G}_{Z_1}(U) + \mathcal{G}_{Z_2}(U) & : U \cap Z_1 \cap Z_2 \neq \emptyset \\ 0 & : U \cap Z_1 \cap Z_2 = \emptyset \end{cases}.$$

Then we have an exact sequence

$$(20) \quad 0 \longrightarrow \mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet) \longrightarrow \prod_{Z \in F^0} \mathcal{G}_Z \xrightarrow{\delta} \prod_{\{Z_1, Z_2\} \in F^1} \mathcal{G}_{\{Z_1, Z_2\}} \longrightarrow 0$$

where δ is the product of all maps $\mathrm{sg}(Z_1). \mathrm{id} : \mathcal{G}_{Z_1} \rightarrow \mathcal{G}_{\{Z_1, Z_2\}}$. In cohomology we get

$$\frac{H^0(\widetilde{\mathfrak{X}}, \prod_{\{Z_1, Z_2\} \in F^1} \mathcal{G}_{\{Z_1, Z_2\}})}{H^0(\widetilde{\mathfrak{X}}, \prod_{Z \in F^0} \mathcal{G}_Z)} \cong H^1(\widetilde{\mathfrak{X}}, \mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet)).$$

We claim that

$$Z_{har}^1(k+2) \otimes \varepsilon^{k+1} \rightarrow H^0(\widetilde{\mathfrak{X}}, \prod_{\{Z_1, Z_2\} \in F^1} \mathcal{G}_{\{Z_1, Z_2\}})$$

$$(f_{\{Z_1, Z_2\}})_{\{Z_1, Z_2\}} \mapsto \prod_{\{Z_1, Z_2\}} \sigma(f_{\{Z_1, Z_2\}})$$

induces an isomorphism $\theta : Z_{har}^1(k+2) \otimes \varepsilon^{k+1} \rightarrow H^1(\widetilde{\mathfrak{X}}, \mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet))$. Since $H^1(\widetilde{\mathfrak{X}}, \mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet)) = H^1(\widetilde{\mathfrak{X}}, \mathcal{O}_{\widehat{\mathfrak{X}}}(-k))$ is flat it suffices to show that the induced map

$$\widetilde{\theta} = \theta/(\widehat{\pi}) : \widetilde{Z}_{har}^1(k+2) \longrightarrow H^1(\widetilde{\mathfrak{X}}, (\mathcal{H}^0(\mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet))_1)$$

is an isomorphism (with notations from the proof of 5.1). Let us first assume $k > 0$ is even. Consider the submodule

$$\widetilde{Z}_{har}^1(k+2)(1) = \left\{ \begin{array}{l} f = (f_{\{Z_1, Z_2\}})_{\{Z_1, Z_2\} \in F^1} \in \widetilde{Z}_{har}^1(k+2); \\ (\gamma \cdot f)_{\{Z_{\gamma_0}, Z_{\gamma_1}\}}(X^{\frac{k}{2}} Y^{\frac{k}{2}}) = 0 \text{ for all } \gamma \in G \end{array} \right\}$$

of $\widetilde{Z}_{har}^1(k+2)$ (this is nothing but the image of $H^0(\widetilde{\mathfrak{X}}, \mathcal{O}_{\widehat{\mathfrak{X}}}(k+2)(1))$ under \widetilde{Res}^0). If for $Z \in F^0$ we let $\widetilde{Z}_{har}^1(k+2)(1)_Z$ be the image of $\widetilde{Z}_{har}^1(k+2)(1) \rightarrow \widetilde{Z}_{har}^1(k+2) \rightarrow \widetilde{Z}_{har}^1(k+2)_Z$, then

$$\widetilde{Z}_{har}^1(k+2)(1) = \prod_{Z \in F^0} \widetilde{Z}_{har}^1(k+2)(1)_Z.$$

In particular we have natural injections $\iota_Z : \tilde{Z}_{har}^1(k+2)(1)_Z \rightarrow \tilde{Z}_{har}^1(k+2)(1)$. We claim that for each $Z \in F^0$ the composition

$$\tilde{Z}_{har}^1(k+2)(1)_Z \xrightarrow{\iota_Z} \tilde{Z}_{har}^1(k+2)(1) \xrightarrow{\tilde{\theta}} H^1(\tilde{\mathfrak{X}}, (\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1) \longrightarrow H^1(\tilde{\mathfrak{X}}, (\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^Z),$$

which we denote by β_Z , is an isomorphism. To see this we may assume $Z = Z_{\gamma_0}$. From the proof of 5.1 we infer an exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^1, \mathcal{H}) \longrightarrow H^0(\mathbb{P}^1, \frac{\mathcal{H}}{\iota^{-1}(\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^{Z_{\gamma_0}}}) \longrightarrow H^1(\tilde{\mathfrak{X}}, (\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^{Z_{\gamma_0}}) \longrightarrow 0.$$

Here $\iota : \mathbb{P}^1 \cong Z_{\gamma_0} \rightarrow \tilde{\mathfrak{X}}$ is the natural embedding, \mathcal{H} is the constant sheaf with value $\oplus_{i=0}^k \mathbb{F}.z^i$, and the quotient $\mathcal{H}/\iota^{-1}(\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^{Z_{\gamma_0}}$ is a skyscraper sheaf whose only stalks are $\frac{k}{2}$ -dimensional \mathbb{F} -vector spaces at the \mathbb{F} -rational points of \mathbb{P}^1 . Namely, in notations from section 2, the \mathbb{F} -rational points of $\mathbb{P}^1 \cong Z_{\gamma_0}$ are just the intersections $Z_{\gamma_0} \cap Z_{\gamma_{a,0}\gamma_{-1}}$ with $a \in R$, and $Z_{\gamma_0} \cap Z_{\gamma_1}$. The stalk of $\mathcal{H}/\iota^{-1}(\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^{Z_{\gamma_0}}$ at $Z_{\gamma_0} \cap Z_{\gamma_{a,0}\gamma_{-1}}$ is (canonically identified with) $\oplus_{i=\frac{k}{2}-1}^k \mathbb{F}.(z-\bar{a})^i$ (with $\bar{a} \in \mathbb{F}$ the image of $a \in R$), and the stalk at $Z_{\gamma_0} \cap Z_{\gamma_1}$ is (canonically identified with) $\oplus_{i=\frac{k}{2}+1}^k \mathbb{F}.z^i$. From the proof of 4.2 we get the exact sequence

$$0 \longrightarrow \tilde{Z}_{har}^1(k+2)(1)_{Z_{\gamma_0}} \longrightarrow \oplus_{j=0}^{\frac{k}{2}-1} \mathbb{F}.h_j \times \prod_{a \in R} \oplus_{j=\frac{k}{2}+1}^k \mathbb{F} . (\gamma_{a,0}h_j) \xrightarrow{\Sigma} \oplus_{j=0}^k \mathbb{F}.h_j$$

(the first factor in the middle term is the $\{Z_{\gamma_0}, Z_{\gamma_1}\}$ -component). Now σ maps $\gamma_{a,0}.h_j$ to $\gamma_{a,0}.z^{k-j} = (z-\bar{a})^{k-j}$, hence defines a map

$$\tilde{Z}_{har}^1(k+2)(1)_{Z_{\gamma_0}} \rightarrow H^0(\mathbb{P}^1, \frac{\mathcal{H}}{\iota^{-1}(\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^{Z_{\gamma_0}}})$$

whose composition with the projection to $H^1(\tilde{\mathfrak{X}}, (\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^{Z_{\gamma_0}})$ is an isomorphism: this isomorphism is our $\beta_{Z_{\gamma_0}}$. We have shown that $\tilde{\theta}|_{\tilde{Z}_{har}^1(k+2)(1)}$ is injective and that its image $\text{Im}(\tilde{\theta}|_{\tilde{Z}_{har}^1(k+2)(1)}) \subset H^1(\tilde{\mathfrak{X}}, (\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1)$ maps isomorphically to $H^1(\tilde{\mathfrak{X}}, \prod_{Z \in F^0} (\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^Z)$. From the exact sequence

$$0 \rightarrow H^0(\tilde{\mathfrak{X}}, \prod_{\{Z_1, Z_2\} \in F^1} (\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^{Z_1, Z_2}) \longrightarrow H^1(\tilde{\mathfrak{X}}, (\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1) \longrightarrow H^1(\tilde{\mathfrak{X}}, \prod_{Z \in F^0} (\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^Z) \longrightarrow 0$$

we therefore get

$$\frac{H^1(\tilde{\mathfrak{X}}, (\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1)}{\text{Im}(\tilde{\theta}|_{\tilde{Z}_{har}^1(k+2)(1)})} \cong H^0(\tilde{\mathfrak{X}}, \prod_{\{Z_1, Z_2\} \in F^1} (\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^{Z_1, Z_2}).$$

In particular we get a map

$$\frac{\tilde{Z}_{har}^1(k+2)}{\tilde{Z}_{har}^1(k+2)(1)} \longrightarrow H^0(\tilde{\mathfrak{X}}, \prod_{\{Z_1, Z_2\} \in F^1} (\mathcal{H}^0(\mathcal{R}_{\tilde{\mathfrak{X}}}^\bullet))_1^{Z_1, Z_2})$$

induced by $\tilde{\theta}$ and it remains to show that this map is bijective. But this is clear, as both sides can be identified with (13). If $k > 0$ is odd things are easier since there are no terms $(\mathcal{H}^0(\mathcal{R}_{\hat{\mathfrak{X}}}^\bullet))_1^{Z_1, Z_2}$ and we only need to show bijectivity of the maps

$$\tilde{Z}_{har}^1(k+2)_Z \xrightarrow{\iota_Z} \tilde{Z}_{har}^1(k+2) \xrightarrow{\tilde{\theta}} H^1(\tilde{\mathfrak{X}}, (\mathcal{H}^0(\mathcal{R}_{\hat{\mathfrak{X}}}^\bullet))_1) \longrightarrow H^1(\tilde{\mathfrak{X}}, (\mathcal{H}^0(\mathcal{R}_{\hat{\mathfrak{X}}}^\bullet))_1^Z).$$

We can proceed just as before, now the sums in our local analysis run from 0 to $\frac{k-1}{2}$, resp. from $\frac{k+1}{2}$ to k . \square

At this point we see that by considering integral structures in our automorphic line bundles $\mathcal{O}_X(k)$ on X we obtain genuinely new structures in cohomology. Namely, whereas Theorem 4.2 does have a non-integral counterpart — the isomorphism

$$Res : \frac{\Gamma(X, \mathcal{O}_X(k+2))}{\text{Im}[\Gamma(X, \mathcal{O}_X(-k)) \xrightarrow{\partial^{k+1}} \Gamma(X, \mathcal{O}_X(k+2))]} \cong C_{har}^1(K)$$

from [6] p.97 —, Theorem 5.5 has no non-integral counterpart (in fact $H^1(X, \mathcal{H}^0(\mathcal{R}_X^\bullet)) = 0$). As an application of Theorem 5.5 we get a *global* version of the monodromy operator, as follows. From 4.2, 5.1 and 5.5 we obtain G -equivariant isomorphisms (if $k > 0$)

$$H^0(\tilde{\mathfrak{X}}, \mathcal{O}_{\hat{\mathfrak{X}}}(k+2)) \cong Z_{har}^1(k+2) \cong H^1(\tilde{\mathfrak{X}}, \mathcal{H}^0(\mathcal{R}_{\hat{\mathfrak{X}}}^\bullet)) \otimes \varepsilon^{-k-1} \cong H^1(\tilde{\mathfrak{X}}, \mathcal{O}_{\hat{\mathfrak{X}}}(-k)) \otimes \varepsilon^{-k-1}$$

whose composition we denote by ν .

Definition: The monodromy operator $N : H^1(\tilde{\mathfrak{X}}, \mathcal{R}_{\hat{\mathfrak{X}}}^\bullet) \rightarrow H^1(\tilde{\mathfrak{X}}, \mathcal{R}_{\hat{\mathfrak{X}}}^\bullet)$ is the composition

$$H^1(\tilde{\mathfrak{X}}, \mathcal{R}_{\hat{\mathfrak{X}}}^\bullet) \xrightarrow{pr} H^0(\tilde{\mathfrak{X}}, \mathcal{O}_{\hat{\mathfrak{X}}}(k+2)) \xrightarrow{\nu} H^1(\tilde{\mathfrak{X}}, \mathcal{O}_{\hat{\mathfrak{X}}}(-k)) \xrightarrow{i} H^1(\tilde{\mathfrak{X}}, \mathcal{R}_{\hat{\mathfrak{X}}}^\bullet)$$

where *pr* resp. *i* is the natural projection resp. inclusion in (18).

Thus N is G -equivariant when viewed as a map $H^1(\tilde{\mathfrak{X}}, \mathcal{R}_{\hat{\mathfrak{X}}}^\bullet) \rightarrow H^1(\tilde{\mathfrak{X}}, \mathcal{R}_{\hat{\mathfrak{X}}}^\bullet) \otimes \varepsilon^{-k-1}$. Its monodromy filtration $\text{Ker}(N) = \text{Im}(N) = H^1(\tilde{\mathfrak{X}}, \mathcal{O}_{\hat{\mathfrak{X}}}(-k))$ splits the Hodge filtration $H^0(\tilde{\mathfrak{X}}, \mathcal{O}_{\hat{\mathfrak{X}}}(k+2))$ of $H^1(\tilde{\mathfrak{X}}, \mathcal{R}_{\hat{\mathfrak{X}}}^\bullet)$. Now we restrict our attention to the action by $\text{SL}_2(K)$. If $\Gamma < \text{SL}_2(K)$ is a cocompact discrete torsion free subgroup, we only need to take Γ -invariants and invert p in (18) to obtain the Hodge decomposition

$$H^1(\hat{X}_\Gamma, (\Omega_{\hat{X}}^\bullet \otimes_{\hat{K}} \text{Sym}_{\hat{K}}^k(\text{St}))^\Gamma) = H^0(\tilde{X}_\Gamma, \mathcal{O}_{\hat{X}}(k+2)^\Gamma) \oplus H^1(\Gamma, \text{Sym}_{\hat{K}}^k(\text{St}))$$

from 5.3 (we saw $H^1(\Gamma, \text{Sym}_{\hat{K}}^k(\text{St})) = H^1(\tilde{\mathfrak{X}}, \mathcal{O}_{\hat{\mathfrak{X}}}(-k))^\Gamma \otimes \mathbb{Q}$ in 5.3): no higher Γ -group cohomology is needed for this passage. It is not hard to see that the monodromy operator we thus obtain on $H^1(\hat{X}_\Gamma, (\Omega_{\hat{X}}^\bullet \otimes_{\hat{K}} \text{Sym}_{\hat{K}}^k(\text{St}))^\Gamma)$ is the one predicted by p -adic Hodge theory, using the description of the latter given in [4]. In particular this shows that N respects the integral *de Rham* structures (as opposed to integral Hyodo-Kato cohomology

structures) in $H^1(\widehat{X}_\Gamma, (\Omega_{\widehat{X}}^\bullet \otimes_{\widehat{K}} \text{Sym}_{\widehat{K}}^k(\text{St}))^\Gamma)$, a fact which the general p -adic Hodge theory does not seem to suggest. We so obtain an infinite rank filtered monodromy module over $\mathcal{O}_{\widehat{K}}$ which comprises all the filtered monodromy modules $H^1(\widehat{X}_\Gamma, (\Omega_{\widehat{X}}^\bullet \otimes_{\widehat{K}} \text{Sym}_{\widehat{K}}^k(\text{St}))^\Gamma)$ for the various Γ .

For $k = 0$ we still can define N de Rham integrally as the composition

$$\begin{aligned} H^1(\widetilde{\mathfrak{X}}_\Gamma, (\mathcal{R}_{\widetilde{\mathfrak{X}}}^\bullet)^\Gamma) &\xrightarrow{pr} H^0(\widetilde{\mathfrak{X}}_\Gamma, \mathcal{O}_{\widetilde{\mathfrak{X}}}(2)^\Gamma) \xrightarrow{Res^0} Z_{har}^1(2)^\Gamma \\ &\xrightarrow{\xi} H^0(\widetilde{\mathfrak{X}}, \prod_{\{Z_1, Z_2\}} \mathcal{G}_{\{Z_1, Z_2\}})^\Gamma \xrightarrow{\delta} H^1(\Gamma, H^0(\widetilde{\mathfrak{X}}, \mathcal{H}^0(\mathcal{R}_{\widetilde{\mathfrak{X}}}^\bullet))) \xrightarrow{i} H^1(\widetilde{\mathfrak{X}}_\Gamma, (\mathcal{R}_{\widetilde{\mathfrak{X}}}^\bullet)^\Gamma). \end{aligned}$$

Here sheaves \mathcal{G}_Z and $\mathcal{G}_{\{Z_1, Z_2\}}$ and a map $Z_{har}^1(2) \rightarrow H^0(\widetilde{\mathfrak{X}}, \prod_{\{Z_1, Z_2\}} \mathcal{G}_{\{Z_1, Z_2\}})$ are defined just as in the proof of 5.5, and ξ is the restricted map on Γ -invariants. The map δ is the connecting homomorphism in group cohomology (observe that for $k = 0$ application of $H^0(\widetilde{\mathfrak{X}}, \cdot)$ to the sequence (20) preserves its exactness). Inverting p in the above composition gives the correct N on $H^1(\widehat{X}_\Gamma, \Omega_{\widehat{X}_\Gamma}^\bullet)$ (at least up to sign, see [4]).

6 Complements

(A) Let $\mathcal{R}_{\widetilde{\mathfrak{X}}}^\bullet = \mathcal{R}_{\widehat{\mathfrak{X}}}^\bullet / (\widehat{\pi})$. One can prove the analogs of 5.1 and 5.3 for $\mathcal{R}_{\widetilde{\mathfrak{X}}}^\bullet$, namely:

$$\begin{aligned} H^1(\widetilde{\mathfrak{X}}, \mathcal{R}_{\widetilde{\mathfrak{X}}}^\bullet) &= H^1(\widetilde{\mathfrak{X}}, \mathcal{H}^0(\mathcal{R}_{\widetilde{\mathfrak{X}}}^\bullet)) \oplus H^0(\widetilde{\mathfrak{X}}, \mathcal{O}_{\widetilde{\mathfrak{X}}}(k+2)) \\ H^1(\widetilde{\mathfrak{X}}_\Gamma, (\mathcal{R}_{\widetilde{\mathfrak{X}}}^\bullet)^\Gamma) &= H^1(\widetilde{\mathfrak{X}}_\Gamma, (\mathcal{H}^0(\mathcal{R}_{\widetilde{\mathfrak{X}}}^\bullet))^\Gamma) \oplus H^0(\widetilde{\mathfrak{X}}_\Gamma, \mathcal{O}_{\widetilde{\mathfrak{X}}}(k+2)^\Gamma). \end{aligned}$$

Note that this is *not* obvious from the proof of 5.1, there we did *not* consider $\mathcal{H}^0(\mathcal{R}_{\widetilde{\mathfrak{X}}}^\bullet)$.

(B) Let $k \in \mathbb{Z}$ be even and let $\omega_{\widehat{\mathfrak{X}}/\mathcal{O}_{\widehat{K}}}$ be the logarithmic differential module of the log smooth morphism $\widehat{\mathfrak{X}} \rightarrow \text{Spf}(\mathcal{O}_{\widehat{K}})$: an invertible $\text{PGL}_2(K)$ -equivariant line bundle on $\widehat{\mathfrak{X}}$. We have an $\text{SL}_2(K)$ -equivariant isomorphism

$$\mathcal{O}_{\widehat{\mathfrak{X}}}(k) \cong \omega_{\widehat{\mathfrak{X}}/\mathcal{O}_{\widehat{K}}}^{\frac{k}{2}}, \quad f \mapsto f dz^{\frac{k}{2}}.$$

Now $dz^{\frac{k}{2}}$ is not a generator of $\omega_{\widehat{\mathfrak{X}}/\mathcal{O}_{\widehat{K}}}^{\frac{k}{2}}$, not even a global section of $\omega_{\widehat{\mathfrak{X}}/\mathcal{O}_{\widehat{K}}}^{\frac{k}{2}}$ if $k < 0$. Let $k > 0$ and even. For $a \in \mathcal{O}_K$ the local section $d\log(z - a)$ is a generator of $\omega_{\widehat{\mathfrak{X}}/\mathcal{O}_{\widehat{K}}}$ on an appropriate open formal subscheme of $\widehat{\mathfrak{X}}$. There, the complex $\mathcal{R}_{\widetilde{\mathfrak{X}}}^\bullet$ becomes isomorphic to

$$\begin{aligned} \omega_{\widehat{\mathfrak{X}}/\mathcal{O}_{\widehat{K}}}^{\frac{-k}{2}} &\longrightarrow \omega_{\widehat{\mathfrak{X}}/\mathcal{O}_{\widehat{K}}}^{\frac{k+2}{2}} \\ f d\log(z - a)^{\frac{-k}{2}} &\mapsto (D_a \prod_{j=1}^{\frac{k}{2}} (D_a^2 - j^2) f) d\log(z - a)^{\frac{k+2}{2}} \end{aligned}$$

where $D_a = (z - a)\partial = \frac{(z-a)d}{d(z-a)}$. For the proof you need to show $(z - a)^{\frac{k+2}{2}}\partial^{k+1}(z - a)^{\frac{k}{2}} = D_a \prod_{j=1}^{\frac{k}{2}} (D_a^2 - j^2)$. For this show by induction on n , departing from $D_a = \partial(z - a) - 1$ that $(z - a)^n \partial^n = D_a(D_a - 1) \dots (D_a - n + 1)$ and $\partial^n(z - a)^n = (D_a + n)(D_a + n - 1) \dots (D_a + 1)$. Also note $-D_a = (z - a)^{-1} \frac{d}{d(z-a)^{-1}}$.

(C) For even weights $k \in \mathbb{Z}$ the $\mathcal{O}_{\widehat{\mathfrak{X}}}$ -modules $\mathcal{O}_{\widehat{\mathfrak{X}}}(k)$ are in fact line bundles, and the base extension $K \rightarrow \widehat{K}$ is unnecessary, i.e. everything we did here descends from $\widehat{\mathfrak{X}}$ to \mathfrak{X} . The automorphic action of even weight k in [8] is the one we get by replacing the factor $\chi^k(\gamma)$ with the factor $\det(\gamma)^{\frac{k}{2}}$ in equation (1). All our results carry over to this situation (and in 5.5 no ε^{k+1} -twist is needed). But also if the weight k is odd, if one is willing to restrict the automorphic action on $\mathcal{O}_X(k)$ to a smaller group, the base extension $K \rightarrow \widehat{K}$ can be avoided and one has equivariant integral structures which are even line bundles. Let $G^{\text{even}} = \{\gamma \in G; \omega(\det(\gamma)) \text{ even}\}$. Note that the restriction to G^{even} of the automorphic action (defined in equation (1)) only depends on the choice of π , not of $\widehat{\pi}$. In notations from section 1, define the following $\mathcal{O}_{\mathfrak{U}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}}$ -submodule of $\mathcal{O}_{\mathfrak{U}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}} \otimes_{\mathcal{O}_K} K$:

$$\mathcal{O}_{\mathfrak{U}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}}(k) = \mathcal{O}_{\mathfrak{U}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}} \cdot f_{n,n}^{\lfloor \frac{kn}{2} \rfloor} f_{n,n+1}^{\lfloor \frac{k(n+1)}{2} \rfloor}.$$

The $\mathcal{O}_{\mathfrak{U}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}}(k)$ glue into an invertible $\mathcal{O}_{\mathfrak{Y}}$ -submodule $\mathcal{O}_{\mathfrak{Y}}(k)$ of $\mathcal{O}_{\mathfrak{Y}} \otimes_{\mathcal{O}_K} K$. Observe

$$\mathcal{O}_{\mathfrak{Y}}(k)|_{\mathfrak{U}_{\{Z_{\gamma_n}\}}} = \pi^{\lfloor \frac{kn}{2} \rfloor} \mathcal{O}_{\mathfrak{U}_{\{Z_{\gamma_n}\}}} \quad \text{inside} \quad \mathcal{O}_{\mathfrak{U}_{\{Z_{\gamma_n}\}}} \otimes_{\mathcal{O}_K} K.$$

As in 1.1 one sees that $\mathcal{O}_{\mathfrak{Y}}(k)$ globalizes to a G^{even} -equivariant line bundle $\mathcal{O}_{\mathfrak{X}}(k)$ on \mathfrak{X} , an integral structure in $\mathcal{O}_X(k)$. Our entire analysis of $\mathcal{O}_{\widehat{\mathfrak{X}}}(k)$ can be repeated with $\mathcal{O}_{\mathfrak{X}}(k)$, with essentially the same results (e.g. those from section 2; however, the case $k = 1$ is slightly harder in this context). One additional feature is that one has to study $\mathcal{O}_{\mathfrak{X}}(k) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_Z$ for $Z \in F_{\text{even}}^0$ and for $Z \in F_{\text{odd}}^0$ separately (the two orbits of G^{even} acting on F^0) and the shapes of these two types are indeed different if k is odd.

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